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Matrix fractions and full system equivalence

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Following the definition by Hayton *et al.* (1990) of full system equivalence for linear multivariable systems, the concept of full unimodular equivalence is defined for matrix fraction descriptions (MFDs). Full unimodular equivalence is an extension of unimodular equivalence, which has been proposed by Kailath (1980) and Smith (1981) for systems described by polynomial matrix models in either left or right MFD form. Full unimodular equivalence has the property of leaving invariant both the finite and infinite structure of the system. It is shown that full unimodular equivalence for MFDs and full system equivalence coincide.

1. Introduction

Consider a linear time-invariant multivariable system Σ described by a polynomial matrix description (PMD):

$$A(\mathsf{D})\boldsymbol{\beta}(t) = B(\mathsf{D})\boldsymbol{u}(t), \qquad y(t) = C(\mathsf{D})\boldsymbol{\beta}(t) + D(\mathsf{D})\boldsymbol{u}(t), \tag{1.1}$$

where D is the (time-) derivative operator, $A(s) \in \mathbb{R}[s]^{r\times r}$ with $\det A(s) \neq 0$, $B(s) \in \mathbb{R}[s]^{r\times m}$, $C(s) \in \mathbb{R}[s]^{p\times r}$, $D(s) \in \mathbb{R}[s]^{p\times m}$, $\boldsymbol{\beta} : [0, \infty) \to \mathbb{R}^r$ is the pseudostate of Σ, $\boldsymbol{u} : [0, \infty) \to \mathbb{R}^m$ is the control input, and $\boldsymbol{y}$ the output of Σ, and let

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)}$$

be the Rosenbrock system matrix of Σ. Then Σ may be written in the form

$$\begin{bmatrix} A(\mathsf{D}) & B(\mathsf{D}) & O \\ -C(\mathsf{D}) & D(\mathsf{D}) & \mathrm{I}_p \\ O & -\mathrm{I}_m & O \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ -\boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}(t) = \begin{bmatrix} O \\ O \\ \mathrm{I}_m \end{bmatrix} \boldsymbol{u}(t), \qquad y(t) = [O, O, \mathrm{I}_p] \begin{bmatrix} \boldsymbol{\beta} \\ -\boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}(t).$$

The polynomial matrix

$$\tilde{P}(s) = \left[\begin{array}{ccc:c} A(s) & B(s) & O & O \\ -C(s) & D(s) & \mathrm{I}_p & O \\ O & -\mathrm{I}_m & O & \mathrm{I}_m \\ \hdashline O & O & -\mathrm{I}_p & O \end{array}\right] =: \left[\begin{array}{c:c} \tilde{T}(s) & \tilde{U} \\ \hdashline -\tilde{V} & O \end{array}\right]$$

is defined by Verghese (1979) as the *normalized form* of the system matrix $P(s)$.

In the particular case when the PMD (1.1), with $r = p$, has the form

$$A(s) = A_L(s) \in \mathbb{R}[s]^{p\times p}, \qquad B(s) = B_L(s) \in \mathbb{R}[s]^{p\times m}, \qquad C(s) = I_p, \qquad D(s) = O_{p,m},$$

the system matrix $P(s)$ is said to be in *left matrix fraction form* and the PMD $[A_L(s), B_L(s), I_p, O_{p,m}]$ is said to be a *left matrix fraction description* of Σ (left MFD). In this form, equations (1.1) correspond to the pair of equations

$$A_L(\mathrm{D})\boldsymbol{\beta}(t) = B_L(\mathrm{D})\boldsymbol{u}(t), \qquad \boldsymbol{y}(t) = \boldsymbol{\beta}(t).$$

Also, in the case when the PMD (1.1), with $r = m$, has the form

$$A(s) = A_R(s) \in \mathbb{R}[s]^{m\times m}, \qquad B(s) = I_m, \qquad C(s) = C_R(s) \in \mathbb{R}[s]^{p\times m}, \qquad D(s) = O_{p,m},$$

the system matrix $P(s)$ is said to be in *right matrix fraction form* and the PMD $[A_R(s), I_m, C_R(s), O_{p,m}]$ is said to be a *right matrix fraction description* of Σ (right MFD). In this form, equations (1.1) correspond to the pair of equations

$$A_R(\mathrm{D})\boldsymbol{\beta}(t) = \boldsymbol{u}(t), \qquad \boldsymbol{y}(t) = C_R(\mathrm{D})\boldsymbol{\beta}(t).$$

Rosenbrock (1970) defined the strict system equivalence (SSE) transformation. This kind of transformation has the property of leaving invariant the finite structure of the system, i.e. the *order* $n := \deg\det A(s)$ of Σ, finite input and output decoupling zeros, finite system zeros and poles of Σ (Rosenbrock 1973, 1974), and the transfer function of Σ: $G(s) = C(s)A^{-1}(s)B(s) + D(s)$. Fuhrmann (1977) later proposed the Fuhrmann system equivalence (FHSE) transformation, a transformation which gives rise to the same equivalence class as that of strict system equivalence. Based on these system equivalence transformations, Kailath (1980) and Smith (1981) defined the unimodular equivalence transformation, for the special case of systems described by a system matrix in left or right matrix fraction form.

Hayton *et al.* (1990) defined the full system equivalence transformation (FSE), a transformation which leaves invariant not only the finite structure of the system Σ, but also its *infinite* structure, i.e. the *generalized order* $f := \delta_M \tilde{T}(s)$ and finite and infinite input and output decoupling zeros (Verghese 1979), finite and infinite system zeros and poles of the system (Ferreira 1980; Walker 1988), and its transfer-function matrix.

In this note, we define the full unimodular equivalence (FUE) transformation for left or right MFDs. This new transformation is an extension of unimodular equivalence transformation since it has the property to leave invariant not only the finite structure of MFDs but also their infinite structure. It is shown that full unimodular equivalence coincides with full system equivalence. It is also shown that all the strongly irreducible polynomial system matrices giving rise to the same transfer function are fully system-equivalent to each other. A relation between left matrix fraction descriptions and right matrix fraction descriptions is also presented. However the main contribution of this paper is to present the special form of full system equivalence in generalized state-space systems and matrix fraction descriptions as in Fig. 1.

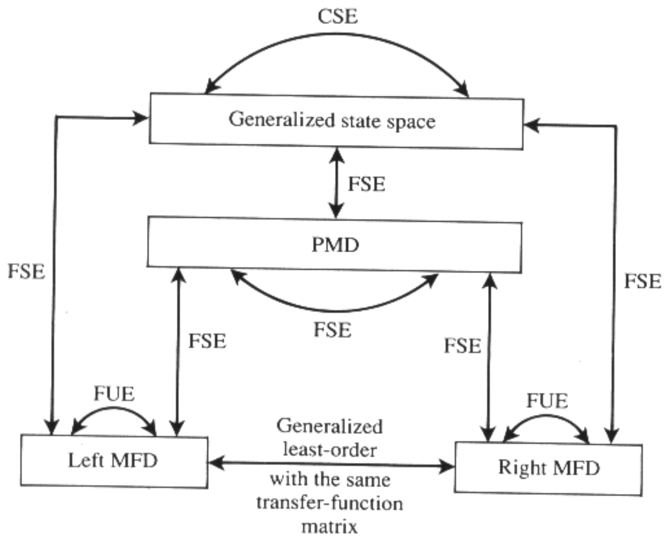

FIG. 1. The special forms of full system equivalence for generalized state-space systems and matrix fraction descriptions (MFDs).

2. Preliminaries and formal definitions

Let $P(s) \in \mathbb{R}[s]^{p \times m}$. The *McMillan degree* of $P(s)$ is the total number of infinite poles of $P(s)$, or equivalently the highest degree of minors of all orders of $P(s)$ (Pugh 1976). Another characterization of $\delta_M P(s)$ is the following and has been noted by Barnett (1971). Let

$$P(s) := P_0 + P_1 s + \cdots + P_q s^q,$$

where $P_i \in \mathbb{R}^{p \times m}$ $(i = 0, \ldots, q)$ with $P_q \neq 0$.

LEMMA 1

$$\delta_M P(s) = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} P_1 & P_2 & \cdots & P_q \\ P_2 & P_3 & \cdots & O \\ \vdots & & \ddots & \vdots \\ P_q & O & \cdots & 0 \end{bmatrix}. \quad \square$$

We shall first define the three classes of system matrices.

DEFINITION 1 Let p and m be fixed positive integers, and r any positive integer. We define the sets

$$\mathcal{R} := \left\{ P(s) = \begin{bmatrix} A_R(s) & I_m \\ -C_R(s) & O_{p,m} \end{bmatrix} : A_R(s) \in \mathbb{R}[s]^{m\times m},\ \mathrm{rank}_{\mathbb{R}[s]} A_R(s) = m, \right.$$

$$\left. C_R(s) \in \mathbb{R}[s]^{p\times m} \right\} \subseteq \mathbb{R}[s]^{(m+p)\times 2m},$$

$$\mathcal{L} := \left\{ P(s) = \begin{bmatrix} A_L(s) & B_L(s) \\ -I_p & O_{p,m} \end{bmatrix} : A_L(s) \in \mathbb{R}[s]^{p\times p},\ \mathrm{rank}_{\mathbb{R}[s]} A_L(s) = p, \right.$$

$$\left. B_L(s) \in \mathbb{R}[s]^{p\times m} \right\} \subseteq \mathbb{R}[s]^{2p\times (p+m)},$$

$$\mathcal{P} := \left\{ P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} : A(s) \in \mathbb{R}[s]^{r\times r},\ \mathrm{rank}_{\mathbb{R}[s]} A(s) = r, \right.$$

$$\left. B(s) \in \mathbb{R}[s]^{r\times m},\ C(s) \in \mathbb{R}[s]^{p\times r},\ D(s) \in \mathbb{R}[s]^{p\times m} \right\} \subseteq \mathbb{R}[s]^{(r\times p)\times (r+m)}. \quad \square$$

Consider also the set $\mathcal{P}(p, m)$ of $(r + p) \times (r + m)$ polynomial matrices, where r is an integer such that $r \geqslant \max\{-p, -m\}$.

Definition 2 (Hayton *et al.* 1988) Two matrices $T_1(s), T_2(s) \in \mathcal{P}(p, m)$ are said to be *fully equivalent* (FE) when there exist polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions such that

$$[M(s),\ T_2(s)] \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = O,$$

where the compound matrices $[M(s), T_2(s)]$ and $\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}$ satisfy the following conditions.

(i) They have full normal rank;
(ii) they have no finite nor infinite zeros;
(iii) the following McMillan-degree conditions hold:

$$\delta_M[M(s),\ T_2(s)] = \delta_M T_2(s), \qquad \delta_M \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = \delta_M T_1(s). \quad \square$$

Let $P(s) \in \mathcal{P}$ be a Rosenbrock system matrix in generalized state-space form. Then the following definition is made.

Definition 3 (Verghese 1979) The operation

$$P(s) = \begin{bmatrix} sE - A & B \\ -C & D \end{bmatrix} \mapsto \begin{bmatrix} N & O \\ X & I \end{bmatrix} \begin{bmatrix} sE - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} N' & Y \\ O & I \end{bmatrix},$$

where N and N' are square nonsingular constant matrices, and X and Y are constant matrices such that

$$XE = O, \qquad EY = O,$$

is termed an *operation of strong equivalence.* □

DEFINITION 4 (Verghese 1979) Two generalized state-space system matrices $P_1(s)$ and $P_2(s)$ (not necessarily of the same dimensions) are said to be *strongly equivalent* if, after some sequence of operations of strong equivalence, and after each system has been trivially deflated as far as possible, the resulting system matrices are related by operations of strong equivalence. □

DEFINITION 5 (Verghese 1979) We define a system

$$\mathcal{S} = \{A(s), B(s), C(s), D(s)\}$$

to be *generalized least-order* or *strongly irreducible* if and only if $\mathcal{S}$ has no finite nor infinite decoupling zeros, or equivalently if and only if the compound matrices

$$\begin{bmatrix} A(s) & B(s) & O \\ -C(s) & D(s) & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \\ O & I \end{bmatrix}$$

have no finite nor infinite zeros. □

An important result which concerns strongly irreducible generalized state-space systems of the same transfer function is the following one.

THEOREM 1 (Verghese *et al.* 1981) Two strongly irreducible systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are strongly equivalent if and only if they have the same transfer function. □

A closed-form expression for strong system equivalence is given by Pugh *et al.* (1987).

DEFINITION 6 (Pugh *et al.* 1987) Let $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$ be two Rosenbrock system matrices in generalized state-space form. $P_1(s)$ and $P_2(s)$ are said to be *completely system equivalent* (CSE) if there exist constant matrices N_1, N_2, X, and Y such that

$$\begin{bmatrix} N_1 & O \\ X & I \end{bmatrix}\begin{bmatrix} sE_1 - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix}\begin{bmatrix} N_2 & Y \\ O & I \end{bmatrix} \tag{2.1}$$

where

$$[N_1, sE_2 - A_2] \quad \text{and} \quad \begin{bmatrix} sE_1 - A_1 \\ -N_2 \end{bmatrix} \quad \text{have no finite or infinite zeros.} \; □ \tag{2.2}$$

Complete system equivalence and strong equivalence do not differ, as we can see in the following result.

THEOREM 2 (Pugh *et al.* 1987) $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$ are strongly equivalent iff they are completely system equivalent. □

Two very useful transformations between polynomial system matrices occur in the following definitions.

DEFINITION 7 (Hayton *et al.* 1990) Let $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$ be two Rosenbrock system matrices. $P_1(s)$ and $P_2(s)$ are said to be *fully system-equivalent* (FSE) if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, and $Y(s)$ such that

$$\begin{bmatrix} M(s) & O \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ O & I \end{bmatrix}, \tag{2.3}$$

where the transformation $P_1(s) \mapsto P_2(s)$ defined by (2.3) is a transformation of full equivalence. □

DEFINITION 8 (Hayton *et al.* 1990) Let $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$ be two Rosenbrock system matrices and let $\tilde{P}_1(s)$ and $\tilde{P}_2(s)$ be respectively their normalized forms. $\tilde{P}_1(s)$ and $\tilde{P}_2(s)$ are said to be *normally fully system-equivalent* (NFSE) if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, and $Y(s)$, such that the corresponding normalized forms $\tilde{P}_1(s)$ and $\tilde{P}_2(s)$ are related by

$$\begin{bmatrix} M(s) & O \\ X(s) & I \end{bmatrix} \begin{bmatrix} \tilde{T}_1(s) & \tilde{U}_1 \\ -\tilde{V}_1 & O \end{bmatrix} = \begin{bmatrix} \tilde{T}_2(s) & \tilde{U}_2 \\ -\tilde{V}_2 & O \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ O & I \end{bmatrix}, \tag{2.4}$$

where the transformation $\tilde{P}_1(s) \mapsto \tilde{P}_2(s)$ defined by (2.14) is a transformation of full equivalence. □

For generalized state-space system matrices, complete system equivalence is the same as full system equivalence as we can see in the following result.

THEOREM 3 (Pugh & Hayton 1990) Two generalized state-space system matrices are completely system-equivalent if and only if they are fully system-equivalent. □

Some useful properties of full system equivalence are the following.

THEOREM 4 (Hayton *et al.* 1990) Under full system equivalence, the following are invariant:

(i) the generalized order f and the Rosenbrock degree d_{R},
(ii) the transfer function and so the finite and infinite transmission poles and zeros,
(iii) the sets of finite and infinite system poles and zeros,
(iv) the sets of finite and infinite input (output) decoupling zeros. □

In this paper we shall study the following transformations.

DEFINITION 9 Given two system matrices

$$P_i(s) = \begin{bmatrix} A_{\mathrm{R}i}(s) & \mathrm{I}_m \\ -C_{\mathrm{R}i}(s) & \mathrm{O}_{p,m} \end{bmatrix} \in \mathcal{R} \quad (i = 1, 2),$$

then $P_1(s)$ and $P_2(s)$ are said to be *fully unimodular equivalent* (FUE) if there exists a unimodular matrix $T_R(s) \in \mathbb{R}[s]^{m \times m}$ such that

$$\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \end{bmatrix} = \begin{bmatrix} A_{R2}(s) & I_m \\ -C_{R2}(s) & O_{p,m} \end{bmatrix} \begin{bmatrix} T_R(s) & O_{m,m} \\ O_{m,m} & I_m \end{bmatrix},$$

$$\begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \\ T_R(s) \end{bmatrix} \text{ has no infinite zeros,} \qquad \delta_M \begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \\ T_R(s) \end{bmatrix} = \delta_M \begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \end{bmatrix}. \quad \square \quad (2.5a, b)$$

DEFINITION 10 Given two system matrices

$$P_i(s) = \begin{bmatrix} A_{Li}(s) & B_{Li}(s) \\ -I_p & O_{p,m} \end{bmatrix} \in \mathcal{L} \quad (i = 1, 2),$$

then $P_1(s)$ and $P_2(s)$ are said to be *fully unimodular equivalent* iff there exists a unimodular matrix $T_L(s) \in \mathbb{R}[s]^{p \times p}$ such that

$$\begin{bmatrix} A_{L1}(s) & B_{L1}(s) \\ -I_p & O_{p,m} \end{bmatrix} = \begin{bmatrix} T_L(s) & O_{m,m} \\ O_{p,p} & I_p \end{bmatrix} \begin{bmatrix} A_{L2}(s) & B_{L2}(s) \\ -I_p & O_{p,m} \end{bmatrix},$$

$[A_{L1}(s), B_{L1}(s), T_L(s)]$ has no infinite zeros,

$$\delta_M[A_{L1}(s), B_{L1}(s), T_L(s)] = \delta_M[A_{L1}(s), B_{L1}(s)]. \quad \square$$

It is proved (Walker 1988) that Definition 7 defines an equivalence relation on $\mathcal{P}$. If $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$ are FSE, then we write $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$. Similarly we write $P_1(s) \overset{\text{FUE}}{\sim} P_2(s)$ when $P_1(s)$ and $P_2(s)$ are FUE in $\mathcal{R}$ (or in $\mathcal{L}$). In the next section, we will show that $\overset{\text{FUE}}{\sim}$ is an equivalence relation which coincides with $\overset{\text{FSE}}{\sim}$ in $\mathcal{R}$ (or in $\mathcal{L}$).

3. Full system equivalence for matrix fraction descriptions

We present some lemmas which will be useful in the sequel.

LEMMA 2 Let $P(s) \in \mathbb{R}[s]^{p \times m}$, and let $Q \in \mathbb{R}^{p \times p}$ be nonsingular. Then

$$\delta_M[QP(s)] = \delta_M P(s).$$

Proof. The result is easily shown by using the Binet–Cauchy formula in the product $Q \cdot P(s)$ and the definition of the McMillan degree of $P(s)$ as the greatest degree of all minors of $P(s)$. $\square$

LEMMA 3 Let $Q(s) \in \mathbb{R}[s]^{p \times m}$, and let $Q_0 \in \mathbb{R}^{p \times m}$, $Q_1 \in \mathbb{R}^{p \times l}$, $Q_2 \in \mathbb{R}^{q \times m}$, and $Q_3 \in \mathbb{R}^{q \times l}$. Then

$$\delta_M \begin{bmatrix} Q(s) + Q_0 & Q_1 \\ Q_2 & Q_3 \end{bmatrix} = \delta_M Q(s).$$

Proof. It can be easily seen from Lemma 1 that the constant terms play no role in the McMillan degree of polynomial matrices, and so the lemma has been proved. $\square$

The following theorem states that full unimodular equivalence of system matrices in right matrix fraction form coincides with full system equivalence.

THEOREM 5 Let

$$P_i(s) = \begin{bmatrix} A_{\mathrm{R}i}(s) & \mathrm{I}_m \\ -C_{\mathrm{R}i}(s) & \mathrm{O}_{p,m} \end{bmatrix} \in \mathcal{R} \quad (i = 1, 2).$$

Then

$$P_1(s) \overset{\text{FUE}}{\sim} P_2(s) \quad \Leftrightarrow \quad P_1(s) \overset{\text{FSE}}{\sim} P_2(s).$$

Proof. We first prove that $P_1(s) \overset{\text{FUE}}{\sim} P_2(s)$ implies $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$. By the definition of full unimodular equivalence, $P_1(s) \overset{\text{FUE}}{\sim} P_2(s)$ implies that (2.5a, b) hold. So

$$\begin{bmatrix} \mathrm{I}_m & \mathrm{O}_{m,p} \\ \mathrm{O}_{p,m} & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} A_{\mathrm{R}1}(s) & \mathrm{I}_m \\ -C_{\mathrm{R}1}(s) & \mathrm{O}_{p,m} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{R}2}(s) & \mathrm{I}_m \\ -C_{\mathrm{R}2}(s) & \mathrm{O}_{p,m} \end{bmatrix} \begin{bmatrix} T_{\mathrm{R}}(s) & \mathrm{O}_{m,m} \\ \mathrm{O}_{m,m} & \mathrm{I}_m \end{bmatrix}.$$

Define the compound matrices

$$\hat{R} = \begin{bmatrix} \mathrm{I}_m & \mathrm{O}_{m,p} & A_{\mathrm{R}2}(s) & \mathrm{I}_m \\ \mathrm{O}_{p,m} & \mathrm{I}_p & -C_{\mathrm{R}2}(s) & \mathrm{O}_{p,m} \end{bmatrix} \quad \text{and} \quad \hat{L} = \begin{bmatrix} A_{\mathrm{R}1}(s) & \mathrm{I}_m \\ -C_{\mathrm{R}1}(s) & \mathrm{O}_{p,m} \\ -T_{\mathrm{R}}(s) & \mathrm{O}_{m,m} \\ \mathrm{O}_{m,m} & -\mathrm{I}_m \end{bmatrix}.$$

(i) $\hat{R}$ has full normal rank because of the unit matrix I_{m+p}. The matrix $\hat{L}$ has also full normal rank because of the unimodular matrix

$$\begin{bmatrix} -T_{\mathrm{R}}(s) & \mathrm{O}_{m,m} \\ \mathrm{O}_{m,m} & -\mathrm{I}_m \end{bmatrix}. \tag{3.1}$$

(ii) $\hat{R}$ has no finite zeros because there exists an $(m + p) \times (m + p)$ minor, i.e. I_{m+p}, with determinant constant. $\hat{L}$ has also no finite zeros because there exists a $2m \times 2m$ minor, i.e. the unimodular matrix (3.1), which has determinant constant.

$\hat{R}$ has no infinite zeros because we can expand the minor of $\hat{R}$ with the greatest degree ($\delta_{\mathrm{M}} \hat{R}$), with unit entries so as to make an $(m + p) \times (m + p)$ minor with the same degree ($\delta_{\mathrm{M}} \hat{R}$), i.e. a condition that guarantees the absence of infinite zeros of a polynomial matrix (Hayton *et al.* 1988).

We also have that

$$\begin{bmatrix} \mathrm{I}_m & O & O & \mathrm{I}_m \\ O & \mathrm{I}_p & O & O \\ O & O & \mathrm{I}_m & O \\ O & O & O & \mathrm{I}_m \end{bmatrix} \hat{L} = \begin{bmatrix} A_{\mathrm{R}1}(s) & \mathrm{O}_{m,m} \\ -C_{\mathrm{R}1}(s) & \mathrm{O}_{p,m} \\ -T_{\mathrm{R}}(s) & \mathrm{O}_{m,m} \\ \mathrm{O}_{m,m} & -\mathrm{I}_m \end{bmatrix}. \tag{3.2}$$

Using (3.2) and (2.5a), we obtain that $\hat{L}$ has no infinite zeros.

(iii) $$\delta_M\begin{bmatrix} I_m & O_{m,p} & A_{R2}(s) & I_m \\ O_{p,m} & I_p & -C_{R2}(s) & O_{p,m}\end{bmatrix} = \delta_M\begin{bmatrix} A_{R2}(s) & I_m \\ -C_{R2}(s) & O_{p,m}\end{bmatrix}$$

by Lemma 3, and a further application of Lemma 3 gives

$$\delta_M\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \\ -T_R(s) & O_{m,m} \\ O_{m,m} & -I_m \end{bmatrix} = \delta_M\begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \\ T_R(s)\end{bmatrix} = \delta_M\begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s)\end{bmatrix} \quad \text{(by (2.5b))}$$

$$= \delta_M\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m}\end{bmatrix} \quad \text{(by Lemma 3).}$$

So finally $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$.

We now prove that $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$ implies $P_1(s) \overset{\text{FUE}}{\sim} P_2(s)$. By the definition of full system equivalence, $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$ implies the existence of $M(s) \in \mathbb{R}[s]^{m\times m}$, $X(s) \in \mathbb{R}[s]^{p\times m}$, $N(s) \in \mathbb{R}[s]^{m\times m}$, and $Y(s) \in \mathbb{R}[s]^{m\times m}$ such that

$$\begin{bmatrix} M(s) & O_{m,p} \\ X(s) & I_p\end{bmatrix}\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m}\end{bmatrix} = \begin{bmatrix} A_{R2}(s) & I_m \\ -C_{R2}(s) & O_{p,m}\end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ O_{m,m} & I_m\end{bmatrix}, \tag{3.3}$$

where the compound matrices

$$\bar{R} = \begin{bmatrix} M(s) & O_{m,p} & A_{R2}(s) & I_m \\ X(s) & I_p & -C_{R2}(s) & O_{p,m}\end{bmatrix} \quad \text{and} \quad \bar{L} = \begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \\ -N(s) & -Y(s) \\ O_{m,m} & -I_m\end{bmatrix}$$

satisfy the conditions:

$$\bar{L} \text{ and } \bar{R} \text{ have full normal rank,} \tag{3.4a}$$

$$\bar{L} \text{ and } \bar{R} \text{ have no finite or infinite zeros,} \tag{3.4b}$$

$$\delta_M\begin{bmatrix} M(s) & O_{m,p} & A_{R2}(s) & I_m \\ X(s) & I_p & -C_{R2}(s) & O_{p,m}\end{bmatrix} = \delta_M\begin{bmatrix} A_{R2}(s) & I_m \\ -C_{R2}(s) & O_{p,m}\end{bmatrix}, \tag{3.4c}$$

$$\delta_M\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \\ -N(s) & -Y(s) \\ O_{m,m} & -I_m\end{bmatrix} = \delta_M\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m}\end{bmatrix}. \tag{3.4d}$$

Now, from (3.3),

$$M(s)A_{R1}(s) = A_{R2}(s)N(s), \tag{3.5a}$$

$$M(s) = A_{R2}(s)Y(s) + I_m, \tag{3.5b}$$

$$X(s)A_{R1}(s) - C_{R1}(s) = -C_{R2}(s)N(s), \tag{3.5c}$$

$$X(s) = -C_{R2}(s)Y(s), \tag{3.5d}$$

Define

$$T_R(s) = N(s) - Y(s)A_{R1}(s) \in \mathbb{R}[s]^{m\times m}. \tag{3.6}$$

Then

$$\begin{aligned} A_{R1}(s) - A_{R2}(s)T_R(s) &= A_{R1}(s) - A_{R2}(s)[N(s) - Y(s)A_{R1}(s)] \\ &= A_{R1}(s) - A_{R2}(s)N(s) + A_{R2}(s)Y(s)A_{R1}(s) \\ &= [I_m - M(s) + A_{R2}(s)Y(s)]A_{R1}(s) \quad \text{(by (3.5a))} \\ &= (I_m - I_m)A_{R1}(s) \quad \text{(by (3.5b))} \\ &= O_{m,m}, \end{aligned}$$

and

$$\begin{aligned} C_{R1}(s) - C_{R2}(s)T_R(s) &= C_{R1}(s) - C_{R2}(s)N(s) + C_{R2}(s)Y(s)A_{R1}(s) \\ &= X(s)A_{R1}(s) + C_{R2}(s)Y(s)A_{R1}(s) \quad \text{(by (3.5c))} \\ &= [X(s) + C_{R2}(s)Y(s)]A_{R1}(s) \\ &= O_{p,m} \quad \text{(by (3.5d))}. \end{aligned}$$

Thus

$$A_{R1}(s) = A_{R2}(s)T_R(s), \qquad C_{R1}(s) = C_{R2}(s)T_R(s). \tag{3.7a,b}$$

Full system equivalence leaves invariant the order n and the finite poles of the system; so

$$\deg\det A_{R1}(s) = \deg\det A_{R2}(s),$$

and from (3.7a) it follows that $\deg\det T_R(s) = 0$ and $\det T_R(s) \in \mathbb{R}\backslash\{0\}$, i.e. $T_R(s)$ is *unimodular*, so we obtain

$$\begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \end{bmatrix} = \begin{bmatrix} A_{R2}(s) & I_m \\ -C_{R2}(s) & O_{p,m} \end{bmatrix} \begin{bmatrix} T_R(s) & O_{m,m} \\ O_{m,m} & I_m \end{bmatrix}. \tag{3.8}$$

Because of (3.4d), we have that $Y(s) =: Y \in \mathbb{R}^{m\times m}$. From (3.6),

$$T_R(s) = N(s) - YA_{R1}(s) \in \mathbb{R}[s]^{m\times m}.$$

Now

$$\begin{bmatrix} A_{R1}(s) & O_{m,m} \\ -C_{R1}(s) & O_{p,m} \\ T_R(s) & O_{m,m} \\ O_{m,m} & I_m \end{bmatrix} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ -Y & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix} \begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \\ -N(s) & -Y \\ O_{m,m} & -I_m \end{bmatrix}. \tag{3.9}$$

Using (3.9) and the FE condition (3.4b), we conclude that

$$\begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \\ T_R(s) \end{bmatrix} \text{ has no infinite zeros.} \tag{3.10}$$

Also, Lemma 2 gives

$$\delta_M \begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \\ T_R(s) \end{bmatrix} = \delta_M \begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \\ -N(s) & -Y \\ O_{m,m} & -I_m \end{bmatrix} = \delta_M \begin{bmatrix} A_{R1}(s) & I_m \\ -C_{R1}(s) & O_{p,m} \end{bmatrix} \quad \text{(by (3.4d))}$$

$$= \delta_M \begin{bmatrix} A_{R1}(s) \\ -C_{R1}(s) \end{bmatrix} \quad \text{(by Lemma 2).} \tag{3.11}$$

So, according to (3.10–11), $P_1(s) \overset{\text{FUE}}{\sim} P_2(s)$. □

We have thus proved that $\overset{\text{FUE}}{\sim} \equiv \overset{\text{FSE}}{\sim}$, i.e. that full system equivalence is an equivalence relation which coincides with full unimodular equivalence for right MFDs. By similar arguments, we can prove the following corresponding result for left MFDs.

THEOREM 6 Let

$$P_i(s) = \begin{bmatrix} A_{Li}(s) & B_{Li}(s) \\ -I_p & O_{p,m} \end{bmatrix} \in \mathcal{L} \quad (i = 1, 2).$$

Then

$$P_1(s) \overset{\text{FUE}}{\sim} P_2(s) \quad \Leftrightarrow \quad P_1(s) \overset{\text{FSE}}{\sim} P_2(s). \quad □$$

As we can easily see from the above theorems, full unimodular equivalence is an extension of unimodular equivalence which was defined by Smith (1981) in the same way as full system equivalence is an extension of Fuhrmann system equivalence, so FUE leaves invariant

(i) the generalized order $f := \delta_M \tilde{T}(s)$ of Σ, and the Rosenbrock degree d_r of Σ,
(ii) the sets of finite and infinite (output) decoupling zeros,

(iii) the transfer function $G(s)$ (finite and infinite transmission poles and zeros),
(iv) the sets of finite and infinite system poles and zeros,

in left (right) matrix fraction descriptions.

Let $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$. Using the reduction method of Bosgra & Van der Weiden (1981), we obtain two fully system-equivalent generalized state-space system matrices $P_{1R}(s)$ and $P_{2R}(s)$ respectively (Hayton *et al.* 1989). We shall show the following.

THEOREM 7 $\quad P_1(s) \overset{\text{FSE}}{\sim} P_2(s) \quad \Leftrightarrow \quad P_{1R}(s) \overset{\text{FSE}}{\sim} P_{2R}(s)$.

Proof. Let $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$, and let $P_{1R}(s)$ and $P_{2R}(s)$ be the corresponding fully system-equivalent generalized state-space system matrices of $P_1(s)$ and $P_2(s)$. This means that

$$P_{1R}(s) \overset{\text{FSE}}{\sim} P_1(s) \overset{\text{FSE}}{\sim} P_2(s) \overset{\text{FSE}}{\sim} P_{2R}(s).$$

So we obtain, by the transitivity property of FSE (Walker 1988), that $P_{1R}(s) \overset{\text{FSE}}{\sim} P_{2R}(s)$.

Let $P_i(s) \in \mathcal{P}$ $(i = 1, 2)$, and let $P_{1R}(s)$ and $P_{2R}(s)$ be the corresponding fully system-equivalent generalized state-space system matrices of $P_1(s)$ and $P_2(s)$. Suppose that $P_{1R}(s) \overset{\text{FSE}}{\sim} P_{2R}(s)$. This means that

$$P_1(s) \overset{\text{FSE}}{\sim} P_{1R}(s) \overset{\text{FSE}}{\sim} P_{2R}(s) \overset{\text{FSE}}{\sim} P_2(s).$$

So, using the transitivity property of FSE, we obtain that $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$. □

THEOREM 8 All strongly irreducible polynomial system matrices giving rise to the same transfer function are fully system equivalent.

Proof. Let $P_1(s)$ and $P_2(s)$ be two strongly irreducible polynomial system matrices, and let $P_{1R}(s)$ and $P_{2R}(s)$ be their fully system-equivalent generalized state-space system matrices (Hayton *et al.* 1989). According to Theorem 7, the only thing we have to show is that $P_{1R}(s) \overset{\text{FSE}}{\sim} P_{2R}(s)$. The generalized state-space system matrices $P_{1R}(s)$ and $P_{2R}(s)$ are strongly irreducible, with the same transfer function matrix because FSE leaves invariant the transfer-function matrix and the finite and infinite zero structure of the corresponding system matrices $P_1(s)$ and $P_2(s)$; so they are strongly equivalent from Theorem 1, or—equivalently from Theorem 2—they are completely system-equivalent. But complete system equivalence implies full system equivalence according to Theorem 3, and so the theorem has been proved. □

To complete this section, we will give a condition for a left MFD and a right MFD to be fully system-equivalent.

THEOREM 9 Let

$$P_i(s) = \begin{bmatrix} A_i(s) & B_i(s) \\ -C_i(s) & D_i(s) \end{bmatrix} \in \mathcal{P} \quad \text{for } i = 1, 2.$$

be generalized least-order. Then

$$P_1(s) \overset{\text{FSE}}{\sim} P_2(s) \quad \Leftrightarrow \quad G_1(s) = G_2(s),$$

where

$$G_i(s) = C_i(s)A_i(s)^{-1}B_i(s) + D_i(s).$$

Proof. ($\Rightarrow$) Since full system equivalence is a special case of Fuhrmann system equivalence, and Fuhrmann system equivalence leaves invariant the transfer function, the same holds for full system equivalence.

($\Leftarrow$) The two PMDs are generalized least-order and so strongly irreducible. Two strongly irreducible PMDs of the same transfer function are FSE according to Theorem 8, and so the converse has been proved.

THEOREM 10 Let

$$P_1(s) = \begin{bmatrix} A_R(s) & I_m \\ -C_R(s) & O_{p,m} \end{bmatrix} \in \mathcal{R} \quad \text{and} \quad P_2(s) = \begin{bmatrix} A_L(s) & B_L(s) \\ -I_p & O_{p,m} \end{bmatrix} \in \mathcal{L}. \tag{3.12}$$

Then

$$P_1(s) \overset{\text{FSE}}{\sim} P_2(s) \quad \Leftrightarrow \quad \begin{cases} B_L(s)A_R(s) = A_L(s)C_R(s), & (3.13\text{a}) \\ [A_L(s), B_L(s)] \text{ and } \begin{bmatrix} A_R(s) \\ -C_R(s) \end{bmatrix} \text{ have no finite or infinite zeros.} & (3.13\text{b}) \end{cases}$$

Proof. ($\Rightarrow$) It is apparent from (3.12) that a necessary condition for $P_1(s)$ and $P_2(s)$ to be FSE is that $P_1(s)$ has no finite or infinite output decoupling zeros and $P_2(s)$ has no finite or infinite input decoupling zeros, or equivalently that $\begin{bmatrix} A_R(s) \\ -C_R(s) \end{bmatrix}$ and $[A_L(s), B_L(s)]$ have no finite or infinite zeros. Hence, from Theorem 9, we require

$$A_L^{-1}(s)B_L(s) = C_R(s)A_R^{-1}(s).$$

($\Leftarrow$) The matrices $[A_L(s), B_L(s)]$ *and* $\begin{bmatrix} A_R(s) \\ -C_R(s) \end{bmatrix}$ have no finite or infinite zeros. So, according to Definition 5, $P_1(s)$ and $P_2(s)$ are generalized least-order. $P_1(s)$ and $P_2(s)$ have also the same transfer function according to (3.13a); hence, according to Theorem 9, we have $P_1(s) \overset{\text{FSE}}{\sim} P_2(s)$. □

4. Conclusions

The full unimodular equivalence transformation has been defined as an extension of the unimodular equivalence transformation. This new transformation leaves invariant the generalized order, the sets of finite and infinite input (output) decoupling zeros, the transfer function, and the sets of finite and infinite system poles and zeros of matrix fraction descriptions. The coincidence of full system equivalence and full unimodular equivalence has also been proved if we consider a left (right) matrix fraction description as a quadruple $[A_L(s), B_L(s), I_p, O_{p,m}]$ ($[A_R(s), I_m, C_R(s), O_{p,m}]$).

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REFERENCES

BARNETT, S. 1971 *Matrices in Control Theory with Applications to Linear Programming.* London: Van Nostrand Reinhold.

BOSGRA, O. H., & VAN DER WEIDEN, A. J. J. 1981 Realizations in generalized state-space form for polynomial system matrices, and the definition of poles, zeros and decoupling zeros at infinity. *Int. J. Control* **33,** 393–411.

FERREIRA, P. M. G. 1980 Infinite system zeros. *Int. J. Control* **32,** 731–5.

FUHRMANN, P. A. 1977 On strict system equivalence and similarity. *Int. J. Control* **25,** 5–10.

HAYTON, G. E., PUGH, A. C., & FRETWELL, P. 1988 Infinite elementary divisors of a matrix polynomial and implications. *Int. J. Control* **47,** 53–64.

HAYTON, G. E., WALKER, A. B., & PUGH, A. C. 1989 Matrix pencil equivalents of a general polynomial matrix. *Int. J. Control* **49,** 979–87.

HAYTON, G. E., WALKER, A. B., & PUGH, A. C. 1991 Infinite frequency structure preserving transformations for general polynomial system matrices. *Int. J. Control* **52,** 1–14.

KAILATH, T. 1980 *Linear Systems.* Englewood Cliffs, N.J.: Prentice-Hall.

PUGH, A. C. 1976 The McMillan degree of a polynomial system matrix. *Int. J. Control* **24,** 129–35.

PUGH, A. C., HAYTON, G. E., & FRETWELL, P. 1987 On transformations of matrix pencils and implications in linear system theory. *Int. J. Control* **45,** 529–48.

PUGH, A. C., & HAYTON, G. E. 1990 On two transformations in the generalized theory of linear systems. *Mathematics Report* A109, Loughborough University, Loughborough, England.

ROSENBROCK, H. H. 1970 *State-Space and Multivariable Theory.* London: Nelson.

ROSENBROCK, H. H. 1973 The zeros of a system. *Int. J. Control* **18,** 297–9.

ROSENBROCK, H. H. 1974 Corrections to "The zeros of a system". *Int. J. Control* **20,** 525–7.

SMITH, M. C. 1981 Matrix fractions and strict system equivalence. *Int. J. Control* **34,** 869–83.

VARDULAKIS, A. I. G. 1991 *Linear Multivariable Control, Algebraic Analysis and Synthesis Methods.* Wiley.

VERGHESE, G. C. 1979 Infinite frequency behaviour in generalized dynamical systems. Ph.D. dissertation, Stanford University, Stanford, CA.

VERGHESE, G. C., LEVY, B. C., & KAILATH, T. 1981 A generalized state-space for singular systems. *IEEE Trans* AC **26,** 811–31.

WALKER, A. B. 1988 Equivalence transformations for linear systems. Ph.D. Thesis, Hull University, Hull, England.