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Generalized state-space system matrix equivalents of a Rosenbrock system matrix

N. P. KARAMPETAKIS AND A. I. VARDULAKIS

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece

Bosgra & Van Der Weiden (1981) have given a procedure whereby a Rosenbrock system matrix may be reduced to an equivalent generalized state-space system matrix. The sense in which this is equivalent to the original system matrix is that the reduced system matrix exhibits identical system properties both at finite and infinite frequencies. Hayton *et al.* (1990) introduced the transformations of *normal full system equivalence* and *full system equivalence*. In the present work, we show that the Bosgra & Van Der Weiden reduction procedure is a *full system-equivalence* transformation, and a characterization of this equivalence in a matrix-transformation sense is also provided.

1. Introduction

Bosgra & Van Der Weiden (1981) have proposed a reduction algorithm which brings a polynomial system matrix to a generalized state-space form under invariance of all relevant system properties, both finite and infinite. Hayton *et al.* (1990) proposed the transformations of *full system equivalence* and *normal full system equivalence*, which leave invariant the finite and infinite zero structure of any polynomial matrix description (PMD) of a linear multivariable system Σ, e.g. its finite and infinite input (output) decoupling zeros, its finite and infinite system zeros and poles, and its finite and infinite transmission zeros and poles.

In the present work, it is shown that the Bosgra & Van Der Weiden reduction of a Rosenbrock system matrix to an equivalent generalized state-space system matrix is a full system equivalence transformation, and a characterization of this equivalence in a matrix-transformation sense is also provided.

Based on full system equivalence, we obtain that *generalized order* (Verghese 1979) as well as input and output dynamical indices are new invariants under full system equivalence of the Rosenbrock system matrix. We observe that the absence of infinite zeros in the denominator matrix of the normalized system matrix reduces the fully system-equivalent *generalized* state-space system matrix of the Rosenbrock system matrix to a fully system-equivalent *regular* state-space system matrix. A relation between the partial states of the two equivalent system matrices is also presented.

2. Preliminary results

Let $P(s) \in \mathbb{R}[s]^{p \times m}$. The *McMillan degree* $\delta_M P(s)$ of $P(s)$ is the total number of infinite poles of $P(s)$, or equivalently the highest degree of minors of all orders of

$P(s)$ (Pugh 1976). Another characterization of $\delta_M P(s)$ is the following and has been noted by Barnett (1971).

Let

$$P(s) = P_0 + P_1 s + P_2 s^2 + \cdots + P_q s^q,$$

where $P_i \in \mathbb{R}^{p\times m}$ $(i = 1, 2, \ldots, q)$ with

$$P_q \neq 0.$$

LEMMA 1 (Barnett 1971; Pugh 1976)

$$\delta_M P(s) = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} P_1 & P_2 & \cdots & P_q \\ P_2 & P_3 & \cdots & O \\ \vdots & & \ddots & \vdots \\ P_q & O & \cdots & O \end{bmatrix}. \quad \square$$

LEMMA 2 (Vardulakis 1991) Let $P(s) \in \mathbb{R}[s]^{r\times r}$, with $\operatorname{rank}_{\mathbb{R}[s]} P(s) = r$ and Smith–McMillan form at $s = \infty$ of $P(s)$, $S^{\infty}_{P(s)}$, given by

$$S^{\infty}_{P(s)} = \operatorname{diag}(s^{q_1}, \ldots, s^{q_v}, 1, \ldots, 1, s^{-\hat{q}_{k+1}}, \ldots, s^{-\hat{q}_r}) \in \mathbb{R}(s)^{r\times r}$$

where $q_1 \geqslant \ldots \geqslant q_v$ and $\hat{q}_{k+1} \leqslant \ldots \leqslant \hat{q}_r$ (all $\geqslant 1$) are respectively the orders of the poles and zeros at $s = \infty$ of $P(s)$, with the middle of the diagonal consisting of $k - v$ 1 s. Then

$$\delta_M P(s) = \sum_{i=1}^{v} q_i = \deg P(s) + \sum_{i=k+1}^{r} \hat{q}_i. \quad \square$$

Consider now a linear time-invariant multivariable system Σ described by a polynomial matrix description (PMD):

$$T(\mathrm{D})\boldsymbol{\xi}(t) = U(\mathrm{D})\boldsymbol{u}(t), \qquad \boldsymbol{y}(t) = V(\mathrm{D})\boldsymbol{\xi}(t) + W(\mathrm{D})\boldsymbol{u}(t),$$

where D is the derivative operator, $T(s) \in \mathbb{R}[s]^{r\times r}$ with $\det T(s) \neq 0$, $U(s) \in \mathbb{R}[s]^{r\times m}$, $V(s) \in \mathbb{R}[s]^{p\times r}$, and $W(s) \in \mathbb{R}[s]^{p\times m}$, and let

$$P(s) := \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \tag{2.1}$$

be the Rosenbrock system matrix of Σ. Then Σ may also be written in the form

$$\begin{bmatrix} T(\mathrm{D}) & U(\mathrm{D}) & O \\ -V(\mathrm{D}) & W(\mathrm{D}) & \mathrm{I}_p \\ O & -\mathrm{I}_m & O \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} O \\ O \\ \mathrm{I}_m \end{bmatrix} u(t), \qquad \boldsymbol{y}(t) = [O, O, \mathrm{I}_p] \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \\ \boldsymbol{y}(t) \end{bmatrix}. \tag{2.2}$$

The polynomial matrix

$$\boldsymbol{P}(s) := \left[\begin{array}{ccc:c} T(s) & U(s) & O & O \\ -V(s) & W(s) & \mathrm{I}_p & O \\ O & -\mathrm{I}_m & O & \mathrm{I}_m \\ \hdashline O & O & -\mathrm{I}_p & O \end{array}\right] =: \begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \end{bmatrix} \tag{2.3}$$

is defined as the *normalized form* of the system matrix $P(s)$ (Verghese 1979).

DEFINITION 1 (Verghese 1979) The number of free response modes of the normalized system (2.2) is termed the *generalized order* f of the system Σ, and is given by

$$f := \delta_{\mathrm{M}}\, \boldsymbol{T}(s). \quad \square$$

REMARK 1 Generalized order is a generalization of the concept of the order n of Σ defined as $n :=$ deg det $T(s)$ (Rosenbrock 1974). It can be shown that $n = f$ if and only if $\boldsymbol{T}(s)$ has no zeros at $s = \infty$ (see Lemma 2). $\square$

DEFINITION 2 (Rosenbrock 1970) The highest degree occurring among all minors of $P(s)$ of the form

$$P, \quad P^{i_1}_{j_1}, \quad P^{i_1 i_2}_{j_1 j_2}, \ldots,$$

where for example $P^{i_1 i_2}_{j_1 j_2}$ is the minor formed from rows $1, \ldots, r, r + i, r + i_2$ and columns $1, \ldots, r, r + j_1, r + j_2$ of $P(s)$, will be called the *Rosenbrock degree* d_r of $P(s)$. $\square$

DEFINITION 3 (Bosgra & Van Der Weiden 1981) The highest degree occurring among all minors of all orders of $P(s)$ will be called the *system degree* d_s of $P(s)$. $\square$

DEFINITION 4 The row-minimal (resp. column-minimal) indices of the compound matrices

$$[\boldsymbol{T}(s), \boldsymbol{U}] \quad \left(\text{resp.} \begin{bmatrix} \boldsymbol{T}(s) \\ -\boldsymbol{V} \end{bmatrix}\right)$$

will be called the *input (*resp. *output) dynamical indices* of the system Σ. $\square$

Consider the set $\mathcal{P}(p, m)$ of $(r + p) \times (r + m)$ polynomial matrices, where the integer r satisfies $r \geqslant \max\{-p, -m\}$.

DEFINITION 5 (Hayton *et al.* 1988) Let $T_1(s), T_2(s) \in \mathcal{P}(p, m)$. Then $T_1(s)$ and $T_2(s)$ are said to be *fully equivalent* (FE) if there exist polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions such that

$$[M(s),\ T_2(s)] \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = O.$$

where

$$[M(s),\ T_2(s)] \text{ and } \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} \text{ have full normal rank and no finite or infinite zeros,} \tag{2.4a}$$

$$\delta_M[M(s),\ T_2(s)] = \delta_M T_2(s), \qquad \delta_M \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = \delta_M T_1(s) \quad \square \tag{2.4b}$$

Hayton *et al.* (1990) have proposed system transformations which preserve the finite and infinite structure of a linear multivariable system Σ.

DEFINITION 6 (Hayton *et al.* 1990) Let $\boldsymbol{P}_1(s)$ and $\boldsymbol{P}_2(s)$ be two polynomial system matrices in normalized form. Then $\boldsymbol{P}_1(s)$ and $\boldsymbol{P}_2(s)$ are said to be *normally fully system-equivalent* (NFSE) if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, and $Y(s)$ such that the corresponding normalized forms $\boldsymbol{P}_1(s)$ and $\boldsymbol{P}_2(s)$ are related by

$$\begin{bmatrix} M(s) & O \\ X(s) & I \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1(s) & \boldsymbol{U}_1 \\ -\boldsymbol{V}_1 & O \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_2(s) & \boldsymbol{U}_2 \\ -\boldsymbol{V}_2 & O \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ O & I \end{bmatrix}, \tag{2.5}$$

where (2.5) is an FE transformation. $\square$

DEFINITION 7 (Hayton *et al.* 1990) Let $P_1(s)$ and $P_2(s)$ be two Rosenbrock system matrices. Then $P_1(s)$ and $P_2(s)$ are said to be *fully system-equivalent* (FSE) if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, and $Y(s)$ such that

$$\begin{bmatrix} M(s) & O \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ O & I \end{bmatrix} \tag{2.6}$$

holds, where (2.6) is an FE transformation. $\square$

LEMMA 3 (Hayton *et al.* 1990) A polynomial system matrix of the form (2.1) and its associated normalized form $\boldsymbol{P}(s)$ in (2.3) are FSE under the full equivalence transformation

$$\begin{bmatrix} \mathrm{I}_r & O \\ O & O \\ O & O \\ O & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

$$= \begin{bmatrix} T(s) & U(s) & O & O \\ -V(s) & W(s) & \mathrm{I}_p & O \\ O & -\mathrm{I}_m & O & \mathrm{I}_m \\ O & O & -\mathrm{I}_p & O \end{bmatrix} \begin{bmatrix} \mathrm{I}_r & O \\ O & \mathrm{I}_m \\ V(s) & -W(s) \\ O & \mathrm{I}_m \end{bmatrix}. \quad \square$$

LEMMA 4 (Hayton *et al.* 1990) Under full system equivalence and normal full system equivalence, the following are invariant:

- the sets of finite and infinite decoupling zeros
- the sets of finite and infinite system and transmission zeros
- the sets of finite and infinite system and transmission poles. □

Based on these new system equivalence transformations, we shall show in the sequel how to reduce a Rosenbrock system matrix to a fully system-equivalent system matrix $P_R(s)$ in generalized state-space form.

3. Reduction of a Rosenbrock system matrix to a fully system-equivalent generalized state-space system matrix

Consider a linear multivariable system represented by a $(r+p)\times(r+m)$ polynomial system matrix

$$P(s) := \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}, \tag{3.1}$$

where det $T(s) \neq 0$, giving rise to the $p \times m$ transfer-function matrix

$$G(s) := V(s)\,T^{-1}(s)\,U(s) + W(s), \tag{3.2}$$

and take its normalized form

$$\boldsymbol{P}(s) := \left[\begin{array}{cccc} T(s) & U(s) & O_{r\times p} & O_{r\times m} \\ -V(s) & W(s) & I_p & O_{p\times m} \\ O_{m\times r} & -I_{m\times m} & O_{m\times p} & I_{m\times m} \\ O_{p\times r} & O_{p\times m} & -I_{p\times p} & O_{p\times m} \end{array}\right] =: \begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \end{bmatrix}. \tag{3.3}$$

Let q be the highest degree occuring among the entries of $P(s)$, and write

$$P(s) := P_0 + P_1 s + P_2 s^2 + \cdots + P_q s^q. \tag{3.4}$$

Define the block Hankel matrices

$$\Pi_E = \begin{bmatrix} P_2 & P_3 & \cdots & P_q \\ P_3 & P_4 & \cdots & O \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ P_q & O & \cdots & O \end{bmatrix}, \qquad \Pi_A = \begin{bmatrix} P_3 & P_4 & \cdots & P_q & O \\ P_4 & P_5 & \cdots & O & O \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ P_q & O & \cdots & O & O \\ O & O & \cdots & O & O \end{bmatrix}, \qquad \Pi_B = \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_q \end{bmatrix},$$

$$\Pi_C = [P_2, P_3, \ldots, P_q],$$

and let $r_E = \operatorname{rank} \Pi_E$. Let $\mathcal{I}$ and $\mathcal{J}$ denote maximal sets of r_E row and column indices such that the rows $\mathcal{I}$ and columns $\mathcal{J}$ of Π_E are linearly independent respectively. Let P_E, P_A, P_B, and P_C be submatrices of Π_E, Π_A, Π_B, and Π_C respectively such that P_E, P_A, and P_B are formed by the rows $\mathcal{I}$ and P_E, P_A, and P_C are formed by the columns $\mathcal{J}$ (see Bosgra & Van Der Weiden (1981)). Define $\bar{r} = r + r_E + p + m$ and the

$(\bar{r}+p) \times (\bar{r}+m)$ system matrix

$$P_{\mathrm{R}}(s) := \left[\begin{array}{cccc} P_1 s + P_0 & P_{\mathrm{C}} s & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} & O \\ P_{\mathrm{B}} s & P_{\mathrm{A}} s - P_{\mathrm{E}} & O & O \\ [O, -\mathrm{I}_m] & O & O & \mathrm{I}_m \\ \hline O & O & -\mathrm{I}_p & O \end{array}\right] =: \begin{bmatrix} \boldsymbol{T}_{\mathrm{R}}(s) & \boldsymbol{U}_{\mathrm{R}} \\ -\boldsymbol{V}_{\mathrm{R}} & O \end{bmatrix}. \tag{3.5}$$

We have the following result.

LEMMA 5 (Bosgra & Van Der Weiden 1981) The matrices $P(s)$ of (3.1) and $P_{\mathrm{R}}(s)$ of (3.5) have the following in common:

- the same transfer function matrix (and thus identical finite and infinite transmission poles and zeros)
- the same Rosenbrock degree d_{r} and system degree d_{s}
- the same decoupling zeros, finite and infinite. □

Before we show that $P(s)$ and $P_{\mathrm{R}}(s)$ are FSE, we shall present some useful lemmas.

LEMMA 6 Let $P(s) \in \mathbb{R}^{p \times m}$, and let $Q \in \mathbb{R}^{p \times p}$ and $N \in \mathbb{R}^{m \times m}$ be nonsingular. Then

$$\delta_{\mathrm{M}}(Q\,P(s)) = \delta_{\mathrm{M}}\,P(s) = \delta_{\mathrm{M}}(P(s)N).$$

Proof. The result is easily shown by using the Binet–Cauchy formula in the product $Q\,P(s)$ and the definition of the McMillan degree of $P(s)$ as the greatest degree of all minors of $P(s)$. □

LEMMA 7 Let $\boldsymbol{T}(s)$ and $\boldsymbol{T}'(s)$ be two FE polynomial matrices, and let P and Q be two square constant nonsingular matrices such that

$$\boldsymbol{T}'(s) = P\,\boldsymbol{T}_{\mathrm{R}}(s)\,Q. \tag{3.6}$$

Then $\boldsymbol{T}(s)$ and $\boldsymbol{T}_{\mathrm{R}}(s)$ are FE.

Proof. $\boldsymbol{T}(s)$ and $\boldsymbol{T}'(s)$ are FE; so there exist polynomial matrices $M(s)$ and $N(s)$ such that

$$M(s)\,\boldsymbol{T}(s) = \boldsymbol{T}'(s)\,N(s),$$

where the compound matrices

$$[M(s),\ \boldsymbol{T}'(s)] \quad \text{and} \quad \begin{bmatrix} \boldsymbol{T}(s) \\ -N(s) \end{bmatrix} \tag{3.7}$$

satisfy the conditions (2.4a,b). Using (3.6), we can see that

$$P^{-1}M(s)\,\boldsymbol{T}(s) = \boldsymbol{T}_{\mathrm{R}}(s)\,Q\,N(s),$$

and the compound matrices

$$[P^{-1}M(s),\ \boldsymbol{T}_{\mathrm{R}}(s)] = P^{-1}[M(s),\ \boldsymbol{T}'(s)]\begin{bmatrix} I & O \\ O & Q^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} \boldsymbol{T}(s) \\ -QN(s) \end{bmatrix} = \begin{bmatrix} I & O \\ O & Q \end{bmatrix}\begin{bmatrix} \boldsymbol{T}(s) \\ -N(s) \end{bmatrix} \tag{3.8}$$

satisfy also the full equivalence conditions (2.4a,b) because of Lemma 6, (3.6), and the full equivalence conditions on the compound matrices (3.7). □

Remark 2. We can also see that $\boldsymbol{T}_{\mathrm{R}}(s) = P^{-1}\boldsymbol{T}'(s)Q^{-1}$. Hence, if $\boldsymbol{T}(s)$ is FE to $\boldsymbol{T}_{\mathrm{R}}(s)$, then $\boldsymbol{T}(s)$ is also FE to $\boldsymbol{T}'(s)$. □

Lemma 8. Let $P(s) \in \mathbb{R}[s]^{p\times m}$, $\boldsymbol{T}_0 \in \mathbb{R}^{p\times m}$, $\boldsymbol{T}_1 \in \mathbb{R}^{p\times l}$, $\boldsymbol{T}_2 \in \mathbb{R}^{q\times m}$, and $\boldsymbol{T}_3 \in \mathbb{R}^{q\times l}$; then

$$\delta_{\mathrm{M}}\begin{bmatrix} P(s) + \boldsymbol{T}_0 & \boldsymbol{T}_1 \\ \boldsymbol{T}_2 & \boldsymbol{T}_3 \end{bmatrix} = \delta_{\mathrm{M}}\, P(s). \tag{3.9}$$

Proof. It can be easily seen from Lemma 1 that the constant terms play no role in the McMillan degree of polynomial matrices, and so the lemma follows. □

Lemma 9 (Hayton *et al.* 1989) If $P(s)$ is an arbitrary polynomial matrix with a corresponding matrix polynomial (3.4), then $P(s)$ is FE to the matrix pencil $P_{\mathrm{F}}(s)$, where

$$P_{\mathrm{F}}(s) = \begin{bmatrix} P_{\mathrm{A}}s - P_{\mathrm{E}} & P_{\mathrm{B}}s \\ P_{\mathrm{C}}s & P_1 s + P_0 \end{bmatrix},$$

and P_{E}, P_{A}, P_{B}, and P_{C} are as previously defined for the reduction of a Rosenbrock system matrix $P(s)$ to an equivalent generalized state-space system matrix. More specifically there exists the following full equivalence transformation between the polynomial matrix $P(s)$ and the pencil $P_{\mathrm{F}}(s)$:

$$\begin{bmatrix} O \\ I \end{bmatrix} P(s) \overset{\mathrm{FE}}{=} \begin{bmatrix} P_{\mathrm{A}}s - P_{\mathrm{E}} & P_{\mathrm{B}}s \\ P_{\mathrm{C}}s & P_1 s + P_0 \end{bmatrix}\begin{bmatrix} (P_{\mathrm{E}} - sP_{\mathrm{A}})^{-1}P_{\mathrm{B}}s \\ I \end{bmatrix}. \quad \square$$

Lemma 10 The polynomial matrices $\boldsymbol{T}(s)$ of (3.3) and $\boldsymbol{T}_{\mathrm{R}}(s)$ of (3.5) are FE.

Proof. We can prove that there exist two polynomial matrices $M(s)$ and $N(s)$ such that

$$M(s)\,\boldsymbol{T}(s) = \boldsymbol{T}_{\mathrm{R}}(s)\,N(s),$$

is a full equivalence transformation. The highest degree occurring among the entries of $\boldsymbol{T}(s)$ is equal to that of $P(s)$, and so we can write

$$\boldsymbol{T}(s) = \boldsymbol{T}_0 + \boldsymbol{T}_1 s + \boldsymbol{T}_2 s^2 + \cdots + \boldsymbol{T}_q s^q$$
$$= \begin{bmatrix} P_0 & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} \\ [O, -\mathrm{I}_m] & O \end{bmatrix} + \begin{bmatrix} P_1 & O \\ O & O \end{bmatrix} s + \cdots + \begin{bmatrix} P_q & O \\ O & O \end{bmatrix} s^q.$$

Define the block Hankel matrices

$$\Pi'_{\mathrm{E}} = \begin{bmatrix} \boldsymbol{T}_2 & \boldsymbol{T}_3 & \cdots & \boldsymbol{T}_q \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 & \cdots & O \\ \vdots & \iddots & \iddots & \vdots \\ \boldsymbol{T}_q & O & \cdots & O \end{bmatrix}, \qquad \Pi'_{\mathrm{A}} = \begin{bmatrix} \boldsymbol{T}_3 & \boldsymbol{T}_4 & \cdots & \boldsymbol{T}_q & O \\ \boldsymbol{T}_4 & \boldsymbol{T}_5 & \cdots & O & O \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \boldsymbol{T}_q & O & \cdots & O & O \\ O & O & \cdots & O & O \end{bmatrix}, \qquad \Pi'_{\mathrm{B}} = \begin{bmatrix} \boldsymbol{T}_2 \\ \boldsymbol{T}_3 \\ \vdots \\ \boldsymbol{T}_q \end{bmatrix},$$
$$\Pi'_{\mathrm{C}} = [\boldsymbol{T}_2, \boldsymbol{T}_3, \ldots, \boldsymbol{T}_q],$$

and let $\rho_{\mathrm{E}} = \operatorname{rank} \Pi'_{\mathrm{E}}$. We can easily see that $r_{\mathrm{E}} = \rho_{\mathrm{E}}$. Let $\mathcal{I}$ and $\mathcal{J}$ denote maximal sets of ρ_{E} row and column indices such that the rows $\mathcal{I}$ and columns $\mathcal{J}$ of Π'_{E} are linearly independent respectively, and let

$$\boldsymbol{T}_{\mathrm{E}} - s\boldsymbol{T}_{\mathrm{A}} = P_{\mathrm{E}} - sP_{\mathrm{A}}, \qquad \boldsymbol{T}_{\mathrm{B}} s = [P_{\mathrm{B}}, O]s, \qquad \boldsymbol{T}_{\mathrm{C}} s = \begin{bmatrix} P_{\mathrm{C}} s \\ O \end{bmatrix}. \tag{3.10}$$

The matrices $\boldsymbol{T}_{\mathrm{E}}$, $\boldsymbol{T}_{\mathrm{A}}$, $\boldsymbol{T}_{\mathrm{B}}$, and $\boldsymbol{T}_{\mathrm{C}}$ are constructed in the same way as P_{E}, P_{A}, P_{B}, and P_{C}, but under the requirement (3.10). According to Lemma 9,

$$\begin{bmatrix} O \\ \mathrm{I}_{r+p+m} \end{bmatrix} \boldsymbol{T}(s)$$
$$\overset{\mathrm{FE}}{\equiv} \begin{bmatrix} \boldsymbol{T}_{\mathrm{A}} s - \boldsymbol{T}_{\mathrm{E}} & \boldsymbol{T}_{\mathrm{B}} s \\ \boldsymbol{T}_{\mathrm{C}} s & \boldsymbol{T}_1 s + \boldsymbol{T}_0 \end{bmatrix} \begin{bmatrix} (\boldsymbol{T}_{\mathrm{E}} - s\boldsymbol{T}_{\mathrm{A}})^{-1} \boldsymbol{T}_{\mathrm{B}} s \\ \mathrm{I}_{r+p+m} \end{bmatrix}$$
$$= \begin{bmatrix} P_{\mathrm{A}} s - P_{\mathrm{E}} & P_{\mathrm{B}} s & O \\ P_{\mathrm{C}} s & P_1 s + P_0 & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} \\ O & [O, -\mathrm{I}_m] & O \end{bmatrix} \begin{bmatrix} (P_{\mathrm{E}} - sP_{\mathrm{A}})^{-1} [P_{\mathrm{B}}, O]s \\ \mathrm{I}_{r+p+m} \end{bmatrix} \quad \text{(by (3.10))}$$
$$= \begin{bmatrix} O & I_{r_{\mathrm{E}}} & O \\ \mathrm{I}_{r+p} & O & O \\ O & O & \mathrm{I}_m \end{bmatrix} \boldsymbol{T}_{\mathrm{R}}(s) \begin{bmatrix} O & I_{r+m} & O \\ \mathrm{I}_{r_{\mathrm{E}}} & O & O \\ O & O & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} (P_{\mathrm{E}} - sP_{\mathrm{A}})^{-1} [P_{\mathrm{B}}, O]s \\ \mathrm{I}_{r+p+m} \end{bmatrix}.$$

So, from Lemma 7, it follows that

$$M\boldsymbol{T}(s) = \boldsymbol{T}_{\mathrm{R}}(s) N(s) \tag{3.11}$$

is a full equivalence transformation, where

$$M=\begin{bmatrix} \mathrm{I}_r & O & O \\ O & \mathrm{I}_p & O \\ O & O & O \\ O & O & \mathrm{I}_m \end{bmatrix}, \qquad N(s)=\left[\begin{array}{c:c} \mathrm{I}_{r+m} & O \\ (P_{\mathrm{E}}-sP_{\mathrm{A}})^{-1}P_{\mathrm{B}}s & O \\ \hdashline O & \mathrm{I}_p \end{array}\right]. \qquad \square$$

THEOREM 1 The polynomial system matrices $\boldsymbol{P}(s)$ of (3.3) and $P_{\mathrm{R}}(s)$ of (3.5) are FSE.

Proof. Using (3.11), it can be easily seen that

$$\begin{bmatrix} M & O \\ O & \mathrm{I}_p \end{bmatrix}\begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \end{bmatrix}=\begin{bmatrix} \boldsymbol{T}_{\mathrm{R}}(s) & \boldsymbol{U}_{\mathrm{R}} \\ -\boldsymbol{V}_{\mathrm{R}} & O \end{bmatrix}\begin{bmatrix} N(s) & O \\ O & \mathrm{I}_m \end{bmatrix}. \tag{3.12}$$

We obtain the compound matrices

$$Q_1(s)=\left[\begin{array}{cc:cc} M & O & \boldsymbol{T}_{\mathrm{R}}(s) & \boldsymbol{U}_{\mathrm{R}} \\ O & \mathrm{I}_p & -\boldsymbol{V}_{\mathrm{R}} & O \end{array}\right] \text{ and } R_1(s)=\left[\begin{array}{cc} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \\ \hdashline -N(s) & O \\ O & -\mathrm{I}_m \end{array}\right] \tag{3.13}$$

and check the full equivalence conditions (2.4a,b)

(i) The polynomial matrix $R_1(s)$ has an $(r+2m+p)\times(r+2m+p)$ unit submatrix which is included in the polynomial matrix

$$\begin{bmatrix} -N(s) & O \\ O & -\mathrm{I}_m \end{bmatrix},$$

and so $R_1(s)$ has full normal rank; no finite zeros, because the greatest common divisor of all $(r+2m+p)\times(r+2m+p)$ minors is equal to 1; and no infinite zeros, because we can always expand the minor of the greatest degree $(\delta_{\mathrm{M}} R_1(s))$ with unit entries so as to incorporate an $(r+2m+p)\times(r+2m+p)$ minor with the same degree—a condition which guarantees the absence of infinite zeros of a polynomial matrix (Hayton *et al.* 1988). We have also that

$$\delta_{\mathrm{M}}\left[\begin{array}{cc} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \\ \hdashline -N(s) & O \\ O & -\mathrm{I}_m \end{array}\right]=\delta_{\mathrm{M}}\begin{bmatrix} \boldsymbol{T}(s) \\ -N(s) \end{bmatrix}=\delta_{\mathrm{M}}\boldsymbol{T}(s)=\delta_{\mathrm{M}}\begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \end{bmatrix},$$

the equalities holding by Lemmas 8, 10, and 8 respectively.

(ii) The polynomial matrix $Q_1(s)$ is of full normal rank because there exists a minor

$$\begin{bmatrix} O & \boldsymbol{T}_{\mathrm{R}}(s) \\ \mathrm{I}_p & -\boldsymbol{V}_{\mathrm{R}} \end{bmatrix}$$

with determinant not equal to zero. We have also that

$$Q_1(s)\begin{bmatrix} O & \mathrm{I}_r & O & O & O & O & O & O & O \\ O & O & \mathrm{I}_p & O & O & O & O & O & O \\ O & O & O & \mathrm{I}_m & O & O & O & O & -\mathrm{I}_m \\ \mathrm{I}_p & O & O & O & O & O & O & \mathrm{I}_p & O \\ O & O & O & O & \mathrm{I}_r & O & O & O & O \\ O & O & O & O & O & \mathrm{I}_m & O & O & O \\ O & O & O & O & O & O & \mathrm{I}_{r_E} & O & O \\ O & O & O & O & O & O & O & \mathrm{I}_p & O \\ O & O & O & O & O & O & O & O & \mathrm{I}_m \end{bmatrix} = \begin{bmatrix} O & M & \boldsymbol{T}_{\mathrm{R}}(s) & O \\ \mathrm{I}_p & O & O & O \end{bmatrix},$$

but we know from the full equivalence conditions of Lemma 10 that the compound matrix $[M, \boldsymbol{T}_{\mathrm{R}}(s)]$ has no finite nor infinite zeros, and so $Q_1(s)$ has also no finite nor infinite zeros. Concerning the McMillan degree conditions, Lemma 8 gives

$$\delta_{\mathrm{M}}\begin{bmatrix} M & O & \boldsymbol{T}_{\mathrm{R}}(s) & \boldsymbol{U}_{\mathrm{R}} \\ O & \mathrm{I}_p & -\boldsymbol{V}_{\mathrm{R}} & O \end{bmatrix} = \delta_{\mathrm{M}}\begin{bmatrix} \boldsymbol{T}_{\mathrm{R}}(s) & \boldsymbol{U}_{\mathrm{R}} \\ -\boldsymbol{V}_{\mathrm{R}} & O \end{bmatrix}. \quad \square$$

It is not only the normalized system matrix which is FSE to the generalized state-space system matrix, but also the Rosenbrock system matrix, as we can see in the following theorem.

THEOREM 2 The Rosenbrock system matrix (3.1) is FSE to the generalized state-space system matrix $P_{\mathrm{R}}(s)$ of (3.5).

Proof. From Lemma 3,

$$\begin{bmatrix} \mathrm{I}_r & O \\ O & O \\ O & O \\ O & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

$$= \begin{bmatrix} T(s) & U(s) & O & O \\ -V(s) & W(s) & \mathrm{I}_p & O \\ O & -\mathrm{I}_m & O & \mathrm{I}_m \\ O & O & -\mathrm{I}_p & O \end{bmatrix} \begin{bmatrix} \mathrm{I}_r & O \\ O & \mathrm{I}_m \\ V(s) & -W(s) \\ O & \mathrm{I}_m \end{bmatrix} \tag{3.14}$$

is a relation of full equivalence. If we relate (3.12) and (3.14), we obtain

$$\begin{bmatrix} I_r & O \\ O & O \\ O & O \\ O & O \\ \hdashline O & I_p \end{bmatrix} \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

$$= \left[\begin{array}{ccc:c} & & & O \\ & T_R(s) & & O \\ & & & I_m \\ \hdashline O & O & -I_p & O \end{array}\right] \left[\begin{array}{c:c} I_r & O \\ 0 & I_m \\ (P_E - sP_A)^{-1}P_B s & \\ V(s) & -W(s) \\ \hdashline O & I_m \end{array}\right]. \tag{3.15}$$

Consider the compound matrices

$$R(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \\ \hdashline -I_r & O \\ O & -I_m \\ -(P_E - sP_A)^{-1}P_B s & \\ -V(s) & W(s) \\ O & -I_m \end{bmatrix}, \tag{3.16a}$$

$$Q(s) = \left[\begin{array}{cc:cccc} I_r & O & P_1 s + P_0 & P_C s & \begin{bmatrix} O \\ I_p \end{bmatrix} & O \\ O & O & & & & \\ O & O & P_B s & P_A s - P_E & O & O \\ O & O & [O, -I_m] & O & O & I_m \\ O & I_p & O & O & -I_p & O \end{array}\right], \tag{3.16b}$$

and check the full equivalence conditions (2.4a,b). We can easily see that the compound matrices $R(s)$ of (3.16a) and $R_1(s)$ of (3.13), as well as the matrix $Q(s)$ of (3.16b) and $Q_1(s)$ of (3.13), can be connected with constant and nonsingular transformations such that

$$\begin{bmatrix} R(s) & O & O \\ O & \mathrm{I}_p & O \\ O & O & \mathrm{I}_m \end{bmatrix} = \begin{bmatrix} \mathrm{I}_r & O & O & O & O & O & O & O & O \\ O & \mathrm{I}_p & O & O & O & O & O & \mathrm{I}_p & O \\ O & O & O & O & \mathrm{I}_r & O & O & O & O \\ O & O & O & O & O & \mathrm{I}_m & O & O & O \\ O & O & O & O & O & O & \mathrm{I}_{r_\mathrm{E}} & O & O \\ O & \mathrm{I}_p & O & \mathrm{I}_p & O & O & O & O & O \\ O & O & \mathrm{I}_m & O & O & O & O & O & \mathrm{I}_m \\ O & O & O & O & O & O & O & -\mathrm{I}_p & O \\ O & O & O & O & O & O & O & O & -\mathrm{I}_m \end{bmatrix} R_1(s), \qquad (3.17a)$$

$$[Q(s), 0] = Q_1(s) \begin{bmatrix} \mathrm{I}_r & O & O & O & O & O & O & O & O \\ O & O & O & O & O & O & O & \mathrm{I}_p & O \\ O & O & O & O & O & O & O & O & \mathrm{I}_m \\ O & \mathrm{I}_p & O & O & O & O & O & \mathrm{I}_p & O \\ O & O & \mathrm{I}_r & O & O & O & O & O & O \\ O & O & O & \mathrm{I}_m & O & O & O & O & O \\ O & O & O & O & \mathrm{I}_{r_\mathrm{E}} & O & O & O & O \\ O & O & O & O & O & \mathrm{I}_p & O & -\mathrm{I}_p & O \\ O & O & O & O & O & O & \mathrm{I}_m & O & -\mathrm{I}_m \end{bmatrix} \qquad (3.17b)$$

Hence, using Lemma 7 and the full equivalence conditions of Theorem 1, as well as (3.17a,b), we can easily see that (3.15) is a full equivalence transformation. □

LEMMA 11 Let $\boldsymbol{T}_\mathrm{E}$, $\boldsymbol{T}_\mathrm{A}$, $\boldsymbol{T}_\mathrm{B}$, and $\boldsymbol{T}_\mathrm{C}$, be constructed as above from the row selection $\mathcal{I}$ and the column selection $\mathcal{J}$. If $\boldsymbol{T}'_\mathrm{E}$, $\boldsymbol{T}'_\mathrm{A}$, $\boldsymbol{T}'_\mathrm{B}$, and $\boldsymbol{T}'_\mathrm{C}$ are formed correspondingly from any other row selection $\mathcal{I}'$ and column selection $\mathcal{J}'$, then there exist constant nonsingular matrices Q_1 and Q_2 such that

$$\boldsymbol{T}_\mathrm{E} - s\boldsymbol{T}_\mathrm{A} = Q_1(\boldsymbol{T}'_\mathrm{E} - s\boldsymbol{T}'_\mathrm{A})Q_2, \qquad \boldsymbol{T}_\mathrm{B} = Q_1\boldsymbol{T}'_\mathrm{B}, \qquad \boldsymbol{T}_\mathrm{C} = \boldsymbol{T}'_\mathrm{C}Q_2,$$

or equivalently such that

$$\begin{bmatrix} I_{r+p} & O & O & O \\ O & Q_1 & O & O \\ O & O & I_m & O \\ O & O & O & I_p \end{bmatrix} \begin{bmatrix} T'_R(s) & U_R \\ -V_R & O \end{bmatrix} \begin{bmatrix} I_{r+m} & O & O & O \\ O & Q_2 & O & O \\ O & O & I_p & O \\ O & O & O & I_m \end{bmatrix}$$

$$= \begin{bmatrix} T_R(s) & U_R \\ -V_R & O \end{bmatrix},$$

where $T'_R(s)$ denotes the denominator matrix of the new generalized state-space system matrix which is the result of the new row and column selection. We conclude therefore that any other generalized state-space realization constructed from different row and column selection is *system-similar* to the previous one. □

4. Useful results on full system equivalence

In this section we present some useful results concerning a Rosenbrock system matrix $P(s)$ (see (3.1)) and the generalized state-space system matrix $P_R(s)$ (see (3.5)) which are FSE.

LEMMA 12 If two Rosenbrock polynomial system matrices $P_1(s)$ and $P_2(s)$ are FSE, then they have the same *generalized order* f and give rise to the same transfer function matrix.

Proof. If we ensure that deg det $T(s)$ = deg det $T(s)$, then the generalised order f_1 of the system Σ_1 (Definition 1) may be written from Lemma 2 as

$$f_1 = \delta_M\, T_1(s) = \sum_{i=1}^{k} q_i(T_1) = \deg T_1(s) + \sum_{i=k+1}^{r} \hat{q}_i(T_1), \tag{4.1}$$

where $q_i(T_1)$ $(i = 1, \dots, k)$ and $\hat{q}_i(T_1)$ $(i = k+1, \dots, \hat{q}_r)$ are respectively the orders of poles and zeros at $s = \infty$ of $T_1(s)$. Hayton *et al.* (1990) have shown (Lemma 4) that full system equivalence preserves the finite and infinite zero structure of the denominator matrices $T_i(s)$ $(i = 1, 2)$ of the two FSE normalized system matrices, so we shall obtain that deg det $T_1(s)$ = deg det $T_2(s)$ and $\hat{q}_i(T_1) = \hat{q}_i(T_2)$. After these conclusions and (4.1), we can easily see that $f_1 = f_2$.

As full system equivalence is a special case of *Fuhrmann system equivalence* (or *extended strict system equivalence*), and Fuhrmann system equivalence leaves invariant the transfer function, the same holds for full system equivalence. □

COROLLARY 1 The generalized state-space system matrix $P_R(s)$ of (3.5) has generalized order

$$f = \deg T(s) + \sum_{i=k+1}^{r+p+m} \hat{q}_i(T),$$

where $\hat{q}_i(\boldsymbol{T})$ $(i = k+1, \ldots, r+p+m)$ are the orders of the zeros at $s = \infty$ of $\boldsymbol{T}(s)$, as will have every other FSE generalized state-space system matrix. □

LEMMA 13 The Rosenbrock degree is invariant under full system equivalence.

Proof. It can be seen from

$$P_{j_1, \ldots, j_q}^{i_1, \ldots, i_q} = G_{j_1, \ldots, j_q}^{i_1, \ldots, i_q} \det T(s), \tag{4.2}$$

where $G_{j_1, \ldots, j_q}^{i_1, \ldots, i_q}$ is the minor formed from rows $i_1, \ldots, i_q$ and columns $j_1, \ldots, j_q$, that the minors of the form

$$P_{j_1, \ldots, j_q}^{i_1, \ldots, i_q}$$

are invariant because the transfer function $G(s)$ and the order n of Σ (Lemma 12) remain invariant under full system equivalence. Since the Rosenbrock degree d_r of Σ is determined by such minors (see (4.2)), it too is invariant. □

LEMMA 14 The generalized order f of Σ (see Definition 1) is equal to the system degree d_s of Σ (see Definition 3).

Proof. By Lemma 8, and consequently by Definition 3,

$$f = \delta_M \begin{bmatrix} T(s) & U(s) & O \\ -V(s) & W(s) & I_p \\ O & -I_m & 0 \end{bmatrix} = \delta_M \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} = d_s. \quad \square$$

Before we show that the input (output) dynamical indices remain invariant from the PMD (3.1) to PMD (3.5), we present a useful lemma.

LEMMA 15 (Verghese 1979) Consider a strongly irreducible normalized system of the form of (2.3).

(i) Let $R(s)$ be a minimal polynomial basis for the right null space of its system matrix $\boldsymbol{P}(s)$, given in (2.3), so that

$$\boldsymbol{P}(s)\, R(s) = O \quad \text{or} \quad \begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & O \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = 0.$$

Then $R_2(s)$ is a minimal polynomial basis for the right null space of $G(s) := \boldsymbol{V}\boldsymbol{T}^{-1}(s)\boldsymbol{U}$, and has the same indices.

(ii) Conversely let $R_2(s)$ be a minimal basis for the right null space of $G(s)$. Then

$$R(s) = \begin{bmatrix} \boldsymbol{T}^{-1}(s)\boldsymbol{U}R_2(s) \\ -R_2(s) \end{bmatrix}$$

is a minimal basis for the right null space of $\boldsymbol{P}(s)$, and has the same indices as $R_2(s)$.

(iii, iv) The respective dual results of (i) and (ii) also hold, relating the left null spaces. □

LEMMA 16 The PMD (3.1) and the PMD (3.5) have the same input/output dynamical indices.

Proof. Consider the compound matrix

$$[\boldsymbol{T}_{\mathrm{R}}(s),\ \boldsymbol{U}_{\mathrm{R}}] = \begin{bmatrix} P_1 s + P_0 & P_{\mathrm{C}} s & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} & O \\ P_{\mathrm{B}} s & P_{\mathrm{A}} s - P_{\mathrm{E}} & O & O \\ [O,\ -\mathrm{I}_m] & O & O & \mathrm{I}_m \end{bmatrix}.$$

This is strictly equivalent to the matrix

$$\hat{\boldsymbol{P}}(s) = \left[\begin{array}{c:ccc} P_{\mathrm{A}} s - P_{\mathrm{E}} & P_{\mathrm{B}} s & O & O \\ \hdashline P_{\mathrm{C}} s & P_1 s + P_0 & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} & O \\ O & [O,\ -\mathrm{I}_m] & O & \mathrm{I}_m \end{array}\right]$$

$$= \begin{bmatrix} O & \mathrm{I}_{r_{\mathrm{E}}} & O \\ \mathrm{I}_{r+p} & O & O \\ O & O & \mathrm{I}_m \end{bmatrix} [\boldsymbol{T}_{\mathrm{R}}(s),\ \boldsymbol{U}_{\mathrm{R}}] \begin{bmatrix} O & \mathrm{I}_{r+m} & O & O \\ \mathrm{I}_{r_{\mathrm{E}}} & O & O & O \\ O & O & \mathrm{I}_p & O \\ O & O & O & \mathrm{I}_m \end{bmatrix},$$

and so $\hat{\boldsymbol{P}}(s)$ and $[\boldsymbol{T}_{\mathrm{R}}(s),\ \boldsymbol{U}_{\mathrm{R}}]$ have the same minimal indices according the properties of strict equivalence transformation. However, we can see that $\hat{\boldsymbol{P}}(s)$ is a Rosenbrock system matrix of a strongly irreducible (according the properties of full equivalence in Lemma 9) generalized state-space system which gives rise to the transfer-function matrix

$$\hat{G}(s) := -\begin{bmatrix} P_{\mathrm{C}} s \\ O \end{bmatrix} (P_{\mathrm{A}} s - P_{\mathrm{E}})^{-1} [P_{\mathrm{B}} s,\ O,\ O] + \begin{bmatrix} P_1 s + P_0 & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} & O \\ [O,\ -\mathrm{I}_m] & O & \mathrm{I}_m \end{bmatrix}$$

$$= \left[\begin{array}{cc:c} P(s) & \begin{bmatrix} O \\ \mathrm{I}_p \end{bmatrix} & O \\ [O,\ -\mathrm{I}_m] & O & \mathrm{I}_m \end{array}\right] = [\boldsymbol{T}(s),\ \boldsymbol{U}].$$

So, according to Lemma 15, the polynomial matrices $[\boldsymbol{T}(s),\ \boldsymbol{U}]$ and $\hat{\boldsymbol{P}}(s)$, or equivalently $[\boldsymbol{T}(s),\ \boldsymbol{U}]$ and $[\boldsymbol{T}_{\mathrm{R}}(s),\ \boldsymbol{U}_{\mathrm{R}}]$, have the same right minimal indices (input dynamical indices). In the same way, we can show that the polynomial matrices $[\boldsymbol{T}(s)^{\mathrm{T}},\ -\boldsymbol{V}^{\mathrm{T}}]^{\mathrm{T}}$ and $[\boldsymbol{T}_{\mathrm{R}}(s)^{\mathrm{T}},\ -\boldsymbol{V}_{\mathrm{R}}^{\mathrm{T}}]^{\mathrm{T}}$ have the same left minimal indices and so the same output dynamical indices. □

THEOREM 3 Full system equivalence leaves invariant the input and output dynamical indices of a system Σ.

Proof. Let $P_1(s)$ and $P_2(s)$ be two FSE Rosenbrock system matrices. Then, according Theorem 2, there exist two FSE generalized state-space system matrices $P_{R1}(s)$ and $P_{R2}(s)$ of $P_1(s)$ and $P_2(s)$ respectively. Full system equivalence is an equivalence relation and so, from its transitivity property, it follows that $P_{R1}(s)$ and $P_{R2}(s)$ are FSE or, equivalently (Pugh & Hayton 1990), completely system-equivalent (i.e. FSE in the case of generalized state-space system matrices). However, complete system equivalence has the property of keeping invariant the input/output dynamical indices (Hayton & Pugh 1985) and this gives that $P_{R1}(s)$ and $P_{R2}(s)$, or equivalently $P_1(s)$ and $P_2(s)$, have the same input/output dynamical indices. □

Full system equivalence leaves invariant (Lemma 4) the finite and infinite decoupling zeros and the finite and infinite system zeros. A consequence of all these properties is the following theorem.

THEOREM 4 The Rosenbrock system matrix $P(s)$ of (3.1) and the generalized state-space system matrix $P_R(s)$ of (3.5) have the same

- generalized order f (or equivalently, from Lemma 14, the system degree d_s) and Rosenbrock degree d_r
- transfer function (and so the same finite and infinite transmission poles and zeros)
- finite and infinite system poles and zeros
- sets of finite and infinite input decoupling zeros
- sets of finite and infinite output decoupling zeros
- input/output dynamical indices.

Proof. The proof is a consequence of Theorem 2, the properties of full system equivalence (see Lemma 4), and Lemmas 12, 13, 14, and 16. □

The dimension of the pseudostate of the FSE generalized state-space system matrix is not arbitrary but is related to the zeros at $\mathbb{C} \cup \{\infty\}$ of the normalized denominator matrix $\boldsymbol{T}(s)$. More specifically, let

$$S^{\infty}_{\boldsymbol{T}(s)} = \operatorname{diag}(s^{q_1}, \ldots, s^{q_v}, 1, \ldots, 1, s^{-\hat{q}_{k+1}}, \ldots, s^{-\hat{q}_{r+p+m}}) \in \mathbb{R}(s)^{(r+p+m)\times(r+p+m)},$$

where $q_1 \geqslant \ldots \geqslant q_v$ and $\hat{q}_{k+1} \leqslant \ldots \leqslant \hat{q}_{r+p+m}$ (all $\geqslant 1$) are respectively the orders of the poles and zeros at $s = \infty$ of $\boldsymbol{T}(s)$, with the middle of the diagonal consisting of $k - v$ 1 s.

THEOREM 5 Consider the Rosenbrock system matrix (3.1) and its FSE (Theorem 2) generalized state-space system matrix $P_R(s)$ in (3.5). Then the dimension μ of the pseudostate of the FSE generalized state-space system matrix (3.5) is given by

$$\mu = \deg T(s) + \sum_{i=v+1}^{r+p+m} (\hat{q}_i + 1),$$

where $\hat{q}_i = 0$ $(i = v + 1, \ldots, k)$.

Proof. The pencil $sE-A$ has order $r+p+m+r_{\mathrm{E}}$, where

$$r_{\mathrm{E}} = \operatorname{rank} \Pi_{\mathrm{E}} = \operatorname{rank} \begin{bmatrix} \boldsymbol{T}_2 & \boldsymbol{T}_3 & \cdots & \boldsymbol{T}_q \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 & \cdots & O \\ \vdots & \vdots & \iddots & \vdots \\ \boldsymbol{T}_q & O & \cdots & O \end{bmatrix} = \delta_{\mathrm{M}}(\boldsymbol{T}_2 s + \boldsymbol{T}_3 s^2 + \cdots + \boldsymbol{T}_q s^{q-1})$$

$$= \delta_{\mathrm{M}} \frac{\boldsymbol{T}(s)}{s} = \sum_{i=1}^{v} q_i(\boldsymbol{T}) - v,$$

and so the dimension μ of the pseudostate of the FSE generalized state-space system matrix (3.5), according to Lemma 2, will be

$$\mu = r+p+m+r_{\mathrm{E}} = r+p+m+\sum_{i=1}^{v} q_i - v = r+p+m+\deg T(s) + \sum_{i=v+1}^{r+p+m} \hat{q}_i - v$$

$$= \deg T(s) + \sum_{i=v+1}^{r+p+m} (\hat{q}_i + 1). \quad \square$$

THEOREM 6 (Vardulakis 1991) If the denominator matrix of the normalized form $\boldsymbol{T}(s)$ has no infinite zeros, then the system matrix has no infinite decoupling zeros. $\square$

COROLLARY 2 Suppose that $\boldsymbol{T}(s)$ has no zeros at $s=\infty$. Then, from the reduction procedure proposed above, and from Theorems 4–6, we obtain an FSE state-space system matrix which has the same finite zero structure, i.e. the same finite system zeros and the same finite decoupling zeros. In other words, when $\boldsymbol{T}(s)$ has no infinite zeros, then full system equivalence and extended strict system equivalence play exactly the same role. $\square$

We shall show now that there exists a relation between the solutions of the two system matrices.

PROPOSITION 1 A relation exists between the pseudostate $\boldsymbol{x}(t)$ of the generalized state-space system matrix (3.5) and the pseudostate $[\boldsymbol{\xi}(t)^{\mathrm{T}}, -\boldsymbol{u}(t)^{\mathrm{T}}]^{\mathrm{T}}$ of the Rosenbrock system matrix (3.1).

Proof. Consider the full system equivalence transformation (3.15):

$$\begin{bmatrix} \mathrm{I}_r & O \\ O & O \\ O & O \\ O & O \\ O & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

$$= \left[\begin{array}{cccc} & & & O \\ & T_{\mathrm{R}}(s) & & O \\ & & & \mathrm{I}_m \\ \hline O & O & -\mathrm{I}_p & O \end{array}\right] \left[\begin{array}{cc} \mathrm{I}_r & O \\ O & \mathrm{I}_m \\ \multicolumn{2}{c}{(P_{\mathrm{E}} - sP_{\mathrm{A}})^{-1} P_{\mathrm{B}} s} \\ V(s) & -W(s) \\ \hline O & \mathrm{I}_m \end{array}\right]. \tag{4.3}$$

If we postmultiply both sides of (4.3) by $[\boldsymbol{\xi}(t)^{\top}, -\boldsymbol{u}(t)^{\top}]^{\top}$, we obtain

$$\left[\begin{array}{cc} \mathrm{I}_r & O \\ O & O \\ O & O \\ O & O \\ \hline O & \mathrm{I}_p \end{array}\right] \begin{bmatrix} T(\mathrm{D}) & U(\mathrm{D}) \\ -V(\mathrm{D}) & W(\mathrm{D}) \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \end{bmatrix}$$

$$= \left[\begin{array}{cccc} & & & O \\ & T_{\mathrm{R}}(\mathrm{D}) & & O \\ & & & \mathrm{I}_m \\ \hline O & O & -\mathrm{I}_p & O \end{array}\right] \left[\begin{array}{cc} \mathrm{I}_r & O \\ 0 & \mathrm{I}_m \\ \multicolumn{2}{c}{(P_{\mathrm{E}} - \mathrm{D}P_{\mathrm{A}})^{-1} P_{\mathrm{B}} \mathrm{D}} \\ V(\mathrm{D}) & -W(\mathrm{D}) \\ \hline O & \mathrm{I}_m \end{array}\right] \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \end{bmatrix} \Leftrightarrow$$

$$\left[\begin{array}{c} \mathbf{0} \\ \hline -\boldsymbol{y}(t) \end{array}\right] = \left[\begin{array}{cccc} & & & O \\ & T_{\mathrm{R}}(\mathrm{D}) & & O \\ & & & \mathrm{I}_m \\ \hline O & O & -\mathrm{I}_p & O \end{array}\right] \left[\begin{array}{cc} \mathrm{I}_r & O \\ 0 & \mathrm{I}_m \\ \multicolumn{2}{c}{(P_{\mathrm{E}} - \mathrm{D}P_{\mathrm{A}})^{-1} P_{\mathrm{B}} \mathrm{D}} \\ V(\mathrm{D}) & -W(\mathrm{D}) \\ \hline O & \mathrm{I}_m \end{array}\right] \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \end{bmatrix},$$

and so

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \\ [(P_{\mathrm{E}} - \mathrm{D}P_{\mathrm{A}})^{-1} P_{\mathrm{B}}]\mathrm{D} \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \end{bmatrix} \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_r & O \\ O & \mathrm{I}_m \\ \multicolumn{2}{c}{[(P_{\mathrm{E}} - \mathrm{D}P_{\mathrm{A}})^{-1} P_{\mathrm{B}}]\mathrm{D}} \\ V(\mathrm{D}) & -W(\mathrm{D}) \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} \mathrm{I}_r & O & O \\ O & \mathrm{I}_m & O \\ \multicolumn{2}{c|}{[(P_{\mathrm{E}} - \mathrm{D}P_{\mathrm{A}})^{-1} P_{\mathrm{B}}]\mathrm{D}} & O \\ O & O & \mathrm{I}_p \end{array}\right] \left[\begin{array}{c} \boldsymbol{\xi}(t) \\ -\boldsymbol{u}(t) \\ \hline \boldsymbol{y}(t) \end{array}\right] \tag{4.4}$$

is a relation between the pseudostate $\boldsymbol{x}(t)$ of the equivalent generalized state-space system matrix (3.5) and the pseudostate $[\boldsymbol{\xi}(t)^{\mathrm{T}}, -\boldsymbol{u}(t)^{\mathrm{T}}, \boldsymbol{y}(t)^{\mathrm{T}}]^{\mathrm{T}}$ of the normalized system matrix (3.3). From (4.4) we can see also that the equation

$$\boldsymbol{\xi}(t) = [\mathrm{I}_r, 0, 0, 0]\,\boldsymbol{x}(t)$$

is a relation between the pseudostate $[\boldsymbol{\xi}(t)^{\mathrm{T}}, -\boldsymbol{u}(t)^{\mathrm{T}}, \boldsymbol{y}(t)^{\mathrm{T}}]^{\mathrm{T}}$ of the normalized system matrix (3.3) and the pseudostate $x(t)$ of the equivalent generalized state-space system matrix (3.5). □

EXAMPLE 1 Let a Rosenbrock system matrix of a system Σ be

$$P(s) := \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} = \begin{bmatrix} s^2+5s+6 & s+1 \\ 2s-5 & 3s+2 \end{bmatrix} \tag{4.5}$$

$(r = p = m = 1)$; equivalently the normalized system matrix is

$$\boldsymbol{P}(s) = \begin{bmatrix} \boldsymbol{T}(s) & \boldsymbol{U} \\ -\boldsymbol{V} & 0 \end{bmatrix} = \left[\begin{array}{ccc:c} s^2+5s+6 & s+1 & 0 & 0 \\ 2s-5 & 3s+2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hdashline 0 & 0 & -1 & 0 \end{array}\right], \tag{4.6}$$

where

$$\boldsymbol{T}(s) = \begin{bmatrix} s^2+5s+6 & s+1 & 0 \\ 2s-5 & 3s+2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

is the denominator matrix of the normalized system matrix of Σ. We can easily find that

$$S^{\infty}_{\boldsymbol{T}(s)} = \operatorname{diag}(s^2, s, s^{-1}), \qquad S^{\mathbb{C}}_{\boldsymbol{T}(s)} = \operatorname{diag}\big(1, 1, (s+2)(s+3)\big).$$

The Rosenbrock system matrix (4.5) can be expanded as

$$P(s) = P_0 + P_1 s + P_2 s^2 =: \begin{bmatrix} 6 & 1 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2.$$

Consider also the following Hankel matrices:

$$\Pi_{\mathrm{E}} := P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Pi_{\mathrm{A}} := \mathrm{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Pi_{\mathrm{B}} := P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Pi_{\mathrm{C}} := P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have that $\operatorname{rank}_{\mathbb{R}} \Pi_{\mathrm{E}} = 1$. Select row 1 and column 1 of Π_{E}. The submatrices P_{E}, P_{A}, P_{B}, and P_{C} for this selection will be

$$P_{\mathrm{E}} = 1, \qquad P_{\mathrm{A}} = 0, \qquad P_{\mathrm{B}} = [1, 0], \qquad P_{\mathrm{C}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and this leads to the system matrix

$$P_R(s) = \left[\begin{array}{cccc} P_1 s + P_0 & P_C s & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 0 \\ P_B s & P_A s - P_E & 0 & 0 \\ [0, -1] & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccccc} 5s+6 & s+1 & s & 0 & 0 \\ 2s-5 & 3s+2 & 0 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]. \tag{4.7}$$

The full system equivalence transformation which relates these two system matrices $\boldsymbol{P}(s)$ and $P_R(s)$ is the following:

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

$$= \left[\begin{array}{cccc} & & & 0 \\ & \boldsymbol{T}_R(s) & & 0 \\ & & & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ (P_E - sP_A)^{-1} P_B s & \\ V(s) & -W(s) \\ \hline 0 & 1 \end{array}\right] \Leftrightarrow$$

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \begin{bmatrix} s^2+5s+6 & s+1 \\ 2s-5 & 3s+2 \end{bmatrix}$$

$$= \left[\begin{array}{ccccc} 5s+6 & s+1 & s & 0 & 0 \\ 2s-5 & 3s+2 & 0 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s & 0 \\ -2s+5 & -3s-2 \\ 0 & 1 \end{bmatrix}.$$

We can see also that the dimension of the pseudostate $\boldsymbol{x}(t)$ of the generalized state-space system is

$$\mu = \deg T(s) + \sum_{i=3}^{3} [\hat{q}_i(\boldsymbol{T}) + 1] = 2 + (1 + 1) = 4.$$

Another useful result is that there exists a relation between the pseudostate $\boldsymbol{x}(t)$ of the generalized state-space system matrix (4.7) and the pseudostate vector $(\xi(t), -u(t), y(t))$ of the normalized system matrix (4.6) of the form

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ [(P_E - \mathrm{D}P_A)^{-1} P_B]\mathrm{D} & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ -u(t) \\ \cdots \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathrm{D} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ -u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \xi(t) \\ -u(t) \\ \dot{\xi}(t) \\ y(t) \end{bmatrix},$$

$\xi(t) = [1, 0, 0, 0]\boldsymbol{x}(t)$. □

5. Conclusions

Bosgra & Van Der Weiden (1981) have given a procedure whereby a Rosenbrock system matrix may be reduced to an equivalent generalized state-space system matrix. The sense in which this is equivalent to the original system matrix is that the reduced system exhibits identical system properties both at finite and infinite frequencies. Hayton *et al.* (1990) introduced the transformations of normal full system equivalence and full system equivalence. In the present work, we have shown that this reduction procedure is a full system equivalence transformation, and a characterization of this equivalence in a matrix transformation has also been provided.

Some useful results of this full system equivalence transformation were presented concerning relations between the properties of the polynomial system matrix and the equivalent generalized state-space system matrix.

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