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On a certain McMillan-degree condition appearing in control

A. C. PUGH*, N. P. KARAMPETAKIS†, G. E. HAYTON‡, AND A. I. VARDULAKIS†
**Department of Mathematical Sciences, Loughborough University, LE11 3TU, U.K.*
†Department of Mathematics, University of Thessaloniki, Thessaloniki 54006, Greece
‡Department of Electronic Engineering, Hull University, Hull, HU6 7RX, U.K.

7 The paper presents a number of interpretations of the McMillan-degree conditions appearing in the transformation of full equivalence defined by Hayton *et al.* in 1988. The most satisfactory explanation seems to be that of guaranteeing the existence of a linear map between the solution sets of the ordinary differential equations which underly the matrices involved in the transformation.

1. Introduction

The generalized theory of linear systems arises from a desire to analyse the point at infinity of the complex plane on the same basis as all other points, since certain systems may exhibit significant behaviour of an impulsive (infinite-frequency) nature. Consequently, when trying to reduce the system equations to some simpler form, it is necessary to ensure that the finite-frequency and infinite-frequency behaviours simultaneously remain invariant. In such a theory therefore, it becomes necessary to determine conditions which guarantee that not only the finite-frequency structure of two polynomial matrices be identical, but also their infinite-frequency structure.

In relation to this problem, [3] and [4] have presented such a set of conditions. The conditions are of three types, two of which are largely expected and a third which is not. This latter condition appears as a restriction on the McMillan degree of certain compound matrices, and no satisfactory explanation of it has yet been given, although two partial explanations were offered in [3] and [7]. Interestingly this McMillan-degree condition has arisen in the work of Zhang [14, 15].

The previous observations underscore the importance of the McMillan-degree condition rather than providing a satisfactory explanation of its role in the above studies, and so the main part of the paper considers this issue. By considering the equivalence to be generated via a map between the solution sets of the linear algebraic and ordinary differential equations underlying the given polynomial matrices, a complete explanation of the McMillan-degree condition can be given.

2. Some conditions guaranteeing identical zero structures

Consider the set $\mathcal{P}(p, m)$ of $(r+p) \times (r+m)$ polynomial matrices, where the integers p and m are fixed, and r satisfies $r \geqslant \max\{-p, -m\}$. A matrix transformation with many important applications in the conventional theory of linear systems and in the context of polynomial models is the following [6, 8].

DEFINITION 2 Two matrices $T_1(s)$ and $T_2(s) \in \mathcal{P}(p,m)$ are said to have extended unimodular equivalence (EUE) if there exist polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions such that

$$M(s)T_1(s) = T_2(s)N(s), \tag{1}$$

where the compound matrices

$$[M(s), T_2(s)] \quad \text{and} \quad \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} \tag{2}$$

(a) have full normal rank and (b) have no finite zeros. □

Important properties of this transformation are stated in the following result.

LEMMA 1 (a) [6] Extended unimodular equivalence defines an equivalence relation on $\mathcal{P}(p,m)$. (b) [6, 8] $T_1(s), T_2(s) \in \mathcal{P}(p,m)$ have EUE iff their Smith forms are related by a trivial expansion. □

It is seen from Lemma 1(b) that the invariant polynomials of the Smith form represent a complete set of independent invariants. In this respect, it is merely the finite-frequency behaviour of the underlying system that is invariant. In the generalized theory of linear systems, it will be necessary to preserve both the finite-zero and infinite-zero structure of polynomial matrices that characterize the finite-frequency and infinite-frequency behaviour of the underlying system, and so the transformation of EUE must be further constrained. The natural expected further restriction is that the component matrices in (2) must additionally have no infinite zeros. However, as can be seen from the following example, these restrictions will not in themselves guarantee the invariance of the finite-zero and infinite-zero structure.

EXAMPLE 1 Let

$$M(s) = \begin{bmatrix} 1 & 2s^2 - s \\ 0 & 1 \end{bmatrix}, \quad T_1(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad T_2(s) = \begin{bmatrix} 1 & s^2 \\ 0 & 1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} 1 & s^2 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Then $M(s)T_1(s) = T_2(s)N(s)$. It is simply verified that the compound matrices (2) formed from (3) have neither finite nor infinite zeros, and that they have full normal rank. However $T_2(s)$ has an infinite zero of order 2, while $T_2(s)$ has one infinite zero of order 1. Thus conditions stating the absence of finite and infinite zeros in the compound matrices (2) will not in themselves guarantee that the transformation (1) preserves the infinite-zero structure. □

The above example indicates that further restrictions must be placed on the compound matrices (2) in order that the transformation resulting from (1) simultaneously preserves the finite-zero and infinite-zero structures.

DEFINITION 2 [3, 4] The matrices $T_1(s), T_2(s) \in \mathcal{P}(p, m)$ are said to be fully equivalent, or to have *full equivalence* (FE), if there exist polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions such that

$$[M(s), T_2(s)]\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = O, \tag{4}$$

where the compound matrices

$$[M(s), T_2(s)] \quad \text{and} \quad \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} \tag{5}$$

(a) have full normal rank; (b) have no finite nor infinite zeros, and (c) satisfy the McMillan-degree conditions

$$\delta_M([M(s), T_2(s)]) = \delta_M(T_2(s)), \qquad \delta_M\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = \delta_M(T_1(s)). \tag{6}$$

□

It is easily seen from (a) and (b) of definition 2, that FE is a special case of the EUE. The further restrictions, which are rather unexpected, are the McMillan-degree conditions (6). In the sequel a number of interpretations will be made concerning these conditions, resulting in a complete explanation of its role in the transformation of FE.

3. Interpretations of the McMillan-degree conditions

One interpretation of the McMillan-degree conditions has been given in [3] and [12] and is stated in the next lemma, in which we use the following notation. Let $T(s) \in \mathcal{P}(p, m)$ be written as

$$T(s) := T_0 + T_1 s + \ldots + T_n s^n$$

where $T_0, \ldots, T_n (T_n \neq 0)$ are constant matrices. The dual of $T(s)$, denoted $D_T(\hat{s})$ is

$$D_T(\hat{s}) := T_0 \hat{s}^n + T_1 \hat{s}^{n-1} + \ldots + T_n \equiv T(1/\hat{s})\hat{s}^n.$$

Parentheses will denote the columnar stacking of matrices; thus

$$(A, B) := \begin{bmatrix} A \\ B \end{bmatrix} \equiv [A^{\mathsf{T}}, B^{\mathsf{T}}]^{\mathsf{T}},$$

where A and B have the same number of columns.

LEMMA 2 [3] Let $D_{(T,\ N)}(\hat{s})$ be the dual of the compound matrix $(T(s), -N(s))$, where $(T(s), -N(s))$ has full normal rank, no infinite zeros and satisfies the McMillan-degree condition

$$\delta_M\left(\begin{bmatrix} T(s) \\ -N(s) \end{bmatrix}\right) = \delta_M(T(s)). \tag{7}$$

If $Q(\hat{s})$ is a nonsingular polynomial matrix such that

$$D_{(T,-N)}(\hat{s}) = \begin{bmatrix} \hat{T}(\hat{s}) \\ -\hat{N}(\hat{s}) \end{bmatrix} Q(\hat{s}), \tag{8}$$

where $(\hat{T}(\hat{s}), -\hat{N}(\hat{s}))$ has no zeros at $\hat{s} = 0$, then the orders of the zeros at the origin of $\hat{T}(\hat{s})$ are precisely the orders of the infinite zeros of $T(s)$. □

The lemma indicates that, under the given conditions, the removal of the greatest common right divisor $Q(\hat{s})$ from the dual of $(T(s), -N(s))$ removes only information about the infinite-pole structure of $T(s)$, and leaves intact the infinite-zero structure. The McMillan-degree condition is vital in this, as the following example demonstrates.

EXAMPLE 2 Let $D_{(T_1,-N)}(\hat{s})$ be the dual of the compound matrix $(T_1(s), -N(s))$ in Example 1. Then

$$D_{(T_1,-N)}(\hat{s}) = \begin{bmatrix} \hat{T}_1(\hat{s}) \\ \hat{N}(\hat{s}) \end{bmatrix} Q(\hat{s}) \iff \begin{bmatrix} \hat{s}^2 & \hat{s} \\ 0 & \hat{s}^2 \\ -\hat{s}^2 & -1 \\ 0 & -\hat{s}^2 \end{bmatrix} = \begin{bmatrix} 1 & \hat{s} \\ 0 & \hat{s}^2 \\ -1 & -1 \\ 0 & -\hat{s}^2 \end{bmatrix} \begin{bmatrix} \hat{s}^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$\hat{T}_1(\hat{s}) = \begin{bmatrix} 1 & \hat{s} \\ 0 & \hat{s}^2 \end{bmatrix}. \tag{9}$$

Now $\hat{T}_1(\hat{s})$ can be seen to possess one zero (i.e. multiplicity 1) at $\hat{s} = 0$ of order 2, which coincides with the multiplicity of the zero at infinity of $T_1(s)$ but not its order (which is 1). It has been seen in Example 1 that the compound matrix $(T_1(s), -N(s))$ has full rank and no infinite zeros. Thus the only reason for the discrepancy between the example and the prediction of Lemma 2 is that the McMillan-degree condition is not satisfied. Note in fact that

$$\delta_{\mathrm{M}}\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = 2 \neq 1 = \delta_{\mathrm{M}}(T_1(s)).$$

It should be noted however that the McMillan-degree condition is satisfied with respect to the matrix $N(s)$, i.e.

$$\delta_{\mathrm{M}}\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = 2 = \delta_{\mathrm{M}}(N(s)).$$

Now note that, in line with Lemma 2, the orders of the zeros at the origin of $\hat{N}(\hat{s})$ coincide exactly with the orders of the infinite zeros of $N(s)$. □

Example 2 demonstrates the importance of the McMillan-degree condition in Lemma 2 since, without this condition, the greatest common right divisor $Q(\hat{s})$ in the example would factor out of the compound matrix $D_{(T_1,-N)}(\hat{s})$ some of the infinite-zero structure of $T_1(s)$. When the condition is satisfied with respect to $N(s)$, then none of the infinite-zero structure of $N(s)$ is lost in the factorization.

A second interpretation of the McMillan-degree condition has been given in [7] and arises from the proof of the following result, which has a slightly weaker hypothesis than that in [7].

LEMMA 3 Let $T_1(s), T_2(s) \in \mathcal{P}(p,m)$ be related via the transformation (4), and let the compound matrices in (5), (a) have full normal rank, (b) have no infinite zeros, and (c) satisfy the McMillan-degree conditions (6); then $T_1(s)$ and $T_2(s)$ have the same infinite-zero structure.

Proof. Let

$$T_2(1/w) = \tilde{D}_2^{-1}(w)\tilde{N}_2(2), \qquad T_1(1/w) = \tilde{N}_1(w)\tilde{D}_1^{-1}(w), \tag{10}$$

be two relatively prime factorizations. Since $T_2(s)$ is polynomial, it has no finite poles—merely infinite ones. Thus $T_2(1/w)$ only has poles at $w = 0$, and so

$$\delta_{\mathrm{M}}(T_2(s)) = \nu(T_2(1/w)) \tag{11}$$

where $\nu(\cdot)$ denotes the least order of the indicated matrix. Further, since the factorizations in (10) are relatively prime, it follows that

$$\delta_{\mathrm{M}}(\det \tilde{D}_2(w)) = \nu(T_2(1/w)). \tag{12}$$

Now suppose that

$$[M(1/w), T_2(1/w)] = D_2^{-1}[N_{21}, N_{22}] \quad \text{and} \quad \begin{bmatrix} T_1(1/w) \\ -N(1/w) \end{bmatrix} = \begin{bmatrix} N_{11} \\ -N_{12} \end{bmatrix} D_1^{-1} \tag{13}$$

are relatively prime factorizations: then

$$\delta_{\mathrm{M}}(\det D_2(w)) = \nu([M(1/w), T_2(1/w)]) = \delta_{\mathrm{M}}([M(s), T_2(s)]) = \delta_{\mathrm{M}}(T_2(s)). \tag{14}$$

It follows from (11) and (14) that

$$\delta_{\mathrm{M}}(\det D_2(w)|) = \nu(T_2(1/w)). \tag{15}$$

Thus $D_2^{-1}(w)N_{22}(w)$ is a prime factorization of $T_2(1/w)$. Therefore $N_{22}(w)$ is a numerator of $T_2(1/w)$, and thus its zero structure at $w = 0$ is the infinite-zero structure of $T_2(s)$.

In an entirely analogous manner, it may be established that $N_{11}(w)$ of (13) is a numerator of $T_1(1/w)$, and that its zero structure at $w = 0$ represents the infinite-zero structure of $T_1(s)$.

Now substituting (13) into (4) gives, on pre- and post-multiplication by $D_2(w)$ and $D_1(w)$ respectively,

$$[N_{21}(w), N_{22}(w)] \begin{bmatrix} N_{11}(w) \\ -N_{12}(w) \end{bmatrix} = O. \tag{16}$$

Now, by condition (b), the compound matrices (5) have no infinite zeros. Accordingly the compound matrices in (16) have full rank at $w = 0$, and so (16) is a local equivalence relation at $s = 0$ [1]. Hence the polynomial matrices $N_{11}(w)$ and $N_{22}(w)$ have the same zero structure at the origin, or equivalently the polynomial matrices $T_1(s)$ and $T_2(s)$ have the same infinite-zero structure. □

Note immediately that the denominator matrices $D_1(w)$ and $D_2(w)$ of (13) have been removed from (16), without affecting the finite-zero structure of $N_{11}(w)$ and $N_{22}(w)$ or equivalently the infinite-zero structure of $T_1(s)$ and $T_2(s)$. This reiterates the comments made following Lemma 2. In fact, Lemma 3 gives an alternative proof of the results of Lemma 2 directly in terms of $T_1(s)$ and $T_2(s)$ rather than their dual forms.

More importantly, however, the proof of Lemma 3 reveals another interpretation of the McMillan-degree conditions. Note that, within the relation (1), or equivalently (4), there are essentially three reasons why the zero structures of $T_1(s)$ and $T_2(s)$ might not coincide. The first reason is that $T_1(s)$ and $N(s)$ for example (similar statements can be made about $M(s)$ and $T_2(s)$) might possess common finite-zero structure, which could therefore be factored out of the relation (4) without ruining the equality but instead ruining the relationship between $T_1(s)$ and $T_2(s)$. It is the condition for the absence of finite zeros in the compound matrix $(T_1(s), -N(s))$ which prevents this occurring, as is apparent in the definition of EUE. Similarly $T_1(s)$ and $N(s)$ might contain common infinite-zero structure which could again be factored out of (4). In the same way, the condition on the absence of infinite zeros in the compound matrix $(T_1(s), -N(s))$ ensures against this happening. The third reason why the zero structures of $T_1(s)$ and $T_2(s)$ might not coincide arises because the relation (4) is based on polynomial matrices which can possess infinite-pole structure. If therefore $N(s)$ had infinite-pole structure in excess of that possessed by $T_1(s)$, then this could be factored out of (4) to leave a matrix $T_1(s)R(s)$ (with $R(s)$ unimodular) in place of $T_1(s)$ which now possessed additional infinite-zero structure by virtue of the factor $R(s)$. It is precisely the McMillan-degree condition which prevents this occurrence. The McMillan-degree conditions are thus seen to be vital to the issue of polynomial-matrix-based transformations that preserve infinite-frequency structure.

EXAMPLE 3 In Example 1, it has been seen that

$$\delta_{\mathrm{M}}([M(s), T_2(s)]) = 2 = \delta_{\mathrm{M}}(T_2(s)) \tag{17}$$

but that

$$\delta_{\mathrm{M}}\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = 2 \neq 1 = \delta_{\mathrm{M}}(T_1(s)). \tag{18}$$

The McMillan-degree conditions of FE are not satisfied. As a result, it is not surprising that the infinite-zero structures of $T_1(s)$ and $T_2(s)$ do not coincide. However, note that the McMillan-degree conditions hold with respect to the matrices $M(s)$ and $N(s)$, i.e.

$$\delta_{\mathrm{M}}([M(s), T_2(s)]) = 2 = \delta_{\mathrm{M}}(M(s)), \tag{19}$$

$$\delta_{\mathrm{M}}\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = 2 = \delta_{\mathrm{M}}(N(s)). \tag{20}$$

Since the compound matrices $[M(s), T_2(s)]$ and $(T_1(s), -N(s))$ have no infinite zeros and full normal rank, it is also not surprising to discover that $M(s)$ and $N(s)$ have the same infinite-zero structure, which confirms Lemma 3. □

In a recent paper, Zhang [14] has presented a necessary and sufficient condition for the existence of a generalized proper rational inverse for nonsquare polynomial matrices, which is subsequently established to be equivalent to the absence of the infinite zeros of a polynomial matrix. A connection exists between this necessary and sufficient condition and the McMillan-degree conditions in definition 2, as will now be described. To establish this connection, let $T(s)$ be a $p \times m$ polynomial matrix, with $p < m$ and $\operatorname{rank}_{\mathbb{R}} T(s) = p$, and let $U(s)$ be any $p \times p$ unimodular matrix such that

$$T(s) = U(s)\bar{T}(s), \tag{21}$$

where $\bar{T}(s)$ is row-proper. Let $\delta_{ri}(\cdot)$ denote the ith row degree of the indicated matrix.

THEOREM 1 [7] With the notation of (21),

$$[U(s), \bar{T}(s)]\begin{bmatrix} T(s) \\ -I \end{bmatrix} = O \tag{22}$$

is a transformation of FE iff

$$\delta_{ri}(U(s)) \leqslant \delta_{ri}(\bar{T}(s)). \tag{23}$$

□

As a result of Theorem 1, the polynomial matrix $T(s)$ is fully equivalent to a polynomial matrix in row-proper form (which therefore has no infinite zeros), and so $T(s)$ has also no infinite zeros. It is thus clear that the row-degree condition (23), which ensures that the polynomial matrix $T(s)$ possesses no infinite zeros, is simply the McMillan-degree condition (6) of FE in the context of the relationship (22).

The above interpretations of the McMillan-degree conditions underscore their use and importance rather than provide a completely satisfactory explanation of their role. To do the latter, we consider a linear homogeneous system of algebraic and ordinary differential equations described by

$$T(\mathsf{D})\boldsymbol{\xi}(t) = \mathbf{0}, \tag{24}$$

where D is the derivative operator (with respect to time) and $T(s) = T_q s^q + \cdots + T_1 q + T_0 \in \mathcal{P}(p,m)$. Define $\mathcal{X}$ as the set of all solutions of (24) corresponding to all possible initial conditions of $\xi(t)$ and its $q-1$ derivatives. Suppose that $\mathcal{X}$ is mapped onto another set $\mathcal{X}_1$ which is defined as the set of all functions $\xi_1(t)$ forming the range of the relation

$$\boldsymbol{\xi}_1(t) = N(\mathsf{D})\boldsymbol{\xi}(t), \tag{25}$$

where $N(s) = N_q s^q + \cdots + N_1 s + N_0 \in \mathcal{P}(p,m)$ is of appropriate dimension and such that one of T_q and N_q is nonzero. In respect of the complete solution space of (24), the relation (25) is not always a map in the formal sense, as can be seen in the following example.

EXAMPLE 4 Consider the homogeneous system of algebraic and differential equations arising from Example 1:

$$\begin{bmatrix} 1 & \mathsf{D} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \mathbf{0}, \tag{26}$$

with solution [9, 10]

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} \xi_2(0-)\delta(t) \\ 0 \end{bmatrix}, \tag{27}$$

where $\delta(\cdot)$ represents a unit impulse at the origin. It seen that the solutions of (26) are determined solely by the initial condition $\xi_2(0-)$. Consider the relation

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & \mathsf{D}^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}. \tag{28}$$

From (27) and (28), we obtain

$$\boldsymbol{y}(t) = \begin{bmatrix} [\xi_2(0-) - \xi_2^{(1)}(0-)]\delta(t) - \xi_2(0-)\delta^{(1)}(t) \\ 0 \end{bmatrix}. \tag{29}$$

It is obvious that relation (28) is not a map in the formal sense because, for any solution $\boldsymbol{\xi}(t)$ of (26), we can obtain many images $\boldsymbol{y}(t)$ which will be determined by the initial conditions $\xi_2(0-)$ and $\xi_2^{(1)}(0-)$. □

The first question to be raised regarding (25) is therefore whether it represents a mapping from $\mathcal{X}$ to $\mathcal{X}_1$ in the formal sense of being a many-one relation. Taking Laplace transforms in (24) and (25) with assumed initial conditions $\boldsymbol{\xi}(0-), \boldsymbol{\xi}^{(1)}(0-), \ldots, \boldsymbol{\xi}^{(q-1)}(0-)$ gives [5]

$$T(s)\tilde{\boldsymbol{\xi}}(s) = \tilde{\boldsymbol{a}}_T(s), \tag{30}$$

$$\tilde{\boldsymbol{\xi}}_1(s) = N(s)\tilde{\boldsymbol{\xi}}(s) - \tilde{\boldsymbol{a}}_N(s), \tag{31}$$

where $\tilde{\boldsymbol{a}}_T(s) = S_{q-1}X_T\bar{\boldsymbol{\xi}}(0-)$, with $\tilde{\boldsymbol{a}}S_{q-1} = [s^{q-1}I, s^{q-2}I, \ldots, I]$ and

$$X_T = \begin{bmatrix} T_q & O & \ldots & O \\ T_{q-1} & T_q & \ldots & O \\ \vdots & & \ddots & \vdots \\ T_1 & T_2 & \ldots & T_q \end{bmatrix}, \qquad \bar{\boldsymbol{\xi}}(0-) = \begin{bmatrix} \boldsymbol{\xi}(0-) \\ \boldsymbol{\xi}^{(1)}(0-) \\ \vdots \\ \boldsymbol{\xi}^{(q-1)}(0-) \end{bmatrix}, \tag{32}$$

and $\tilde{\boldsymbol{a}}_N(s) = S_{q-1}X_N\bar{\boldsymbol{\xi}}(0-)$, with

$$X_N = \begin{bmatrix} N_q & O & \ldots & O \\ N_{q-1} & N_q & \ldots & O \\ \vdots & & \ddots & \vdots \\ N_1 & N_2 & \ldots & N_q \end{bmatrix}. \tag{33}$$

THEOREM 2 The relation (25) is a mapping in the formal sense (many-one relation) if and only if

$$\delta_M\left(\begin{bmatrix} T(s) \\ N(s) \end{bmatrix}\right) = \delta_M(T(s)). \tag{34}$$

Proof. The relation (25) is a mapping in the formal sense iff it uniquely specifies an image $\xi_1(t)$ for each solution $\xi(t)$ of (24). Accordingly, in respect of (30) and (31), the relation (25) is a mapping iff, for each $\hat{\xi}(s)$ determined by (30), the relation (31) determines $\hat{\xi}_1(s)$ uniquely. It is apparent that, if a given $\hat{\xi}(s)$ has two images $\hat{\xi}_1(s)$ and $\hat{\xi}_1'(s)$ under (31), then this is entirely due to the $\hat{a}_N(s)$ vector or (more particularly) the associated initial-condition vector $\bar{\xi}(0-)$ of (32). Let therefore $\bar{\xi}(0-) \neq \bar{\xi}'(0-)$ be two initial condition vectors. It then follows that these determine the same solution $\xi(t)$ provided that $\bar{\xi}(0-) - \bar{\xi}'(0-) \in \ker X_T$. Given that this condition is satisfied, it then follows that $\hat{\xi}_1(s)$ is determined via (31) as the unique image of $\hat{\xi}(s)$, provided that $\bar{\xi}(0-) - \bar{\xi}'(0-) \in \ker X_N$. Hence (25) is a map in the formal sense iff

$$\ker X_T \subseteq \ker X_N \tag{35}$$

or equivalently

$$\ker\begin{bmatrix} X_T \\ X_N \end{bmatrix} = \ker X_T. \tag{36}$$

From the dimension theorem of linear mappings, (36) holds iff

$$\operatorname{rank}_{\mathbb{R}}\begin{bmatrix} X_T \\ X_N \end{bmatrix} = \operatorname{rank}_{\mathbb{R}} X_T. \tag{37}$$

Hence, from the well-known characterization of the McMillan degree [5],

$$\delta_M\left(\begin{bmatrix} T(s) \\ N(s) \end{bmatrix}\right) = \delta_M(T(s)), \tag{38}$$

as required. □

REMARK 1 In Example 4, it follows from Theorem 2, since

$$\delta_M\left(\begin{bmatrix} T(s) \\ N(s) \end{bmatrix}\right) = 2 \neq 1 = \delta_M(T(s)), \tag{39}$$

that relation (28) is not a formal map of the solution space of the system (26). □

It can be further seen from Theorem 2 that the McMillan-degree condition (34) guarantees that the polynomial part of $\hat{\xi}_1(s)$ is uniquely determined via (31) by the particular pseudostate $\hat{\xi}(s)$. It is this which determines that the map (31) is well constituted in the formal sense. It should be emphasized that consideration of the polynomial part of $\hat{\xi}_1(s)$ in relation (31) is not necessary in the case of the kind of mapping which underlies EUE, since only its restriction to the finite-frequency solution space, or equivalently the strictly proper part of $\hat{\xi}_1(s)$, is of interest. In any study of merely the finite-frequency aspects of a system's behaviour, which is the aim of the EUE transformation, the McMillan-degree conditions will play no role. We can now state the next result.

COROLLARY 1 Let $T_1(s), T_2(S) \in \mathcal{P}(p, m)$ be related as

$$M(s)T_1(s) = T_2(s)N(s). \tag{40}$$

Then the McMillan-degree condition

$$\delta_M\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = \delta_M(T_1(s)) \quad (\text{resp.} \quad \delta_M([M(s), T_2(s)]) = \delta_M(T_2(s))) \tag{41}$$

is precisely the requirement for the existence of a linear map between the solution spaces of the homogeneous systems S_1 and S_2 (S_1' and S_2') described by:

$$T_i(\mathsf{D})\boldsymbol{\xi}_i(t) = \mathbf{0} \quad (T_i^{\mathsf{T}}(\mathsf{D})\boldsymbol{\xi}_i'(t) = \mathbf{0}) \quad \text{for } i = 1, 2 \tag{42}$$

Proof. Postmultiplying (premultiplying) both sides of (40) with the pseudostate $\boldsymbol{\xi}_1(t)$ $(\boldsymbol{\xi}_2'(t))$, we obtain

$$T_2(\mathsf{D})[N(\mathsf{D})\boldsymbol{\xi}_1(\mathrm{t})] = 0 \quad (\mathrm{T}_1^{\mathsf{T}}(\mathsf{D})[\mathrm{M}(\mathsf{D})^{\mathsf{T}}\boldsymbol{\xi}_2'(\mathrm{t})] = 0). \tag{43}$$

Hence

$$\boldsymbol{\xi}_2(t) = N(\mathsf{D})\xi_1(t) \quad (\boldsymbol{\xi}_1(t) = M(\mathsf{D})^{\mathsf{T}}\boldsymbol{\xi}_2'(t)) \tag{44}$$

gives a solution of the homogeneous system S_2 (S_1'). The McMillan-degree condition (41) now guarantees that the relation (44) is a linear map between the solution spaces of S_1 and S_2 (S_1' and S_2'). □

REMARK 2 It follows from Corollary 1 and Example 4 that

$$[y_1'(t), y_2'(t)] = [\xi_1'(t), \xi_2'(t)]\begin{bmatrix} 1 & 2\mathsf{D}^2 - \mathsf{D} \\ 0 & 1 \end{bmatrix} := \left(\begin{bmatrix} 1 & 0 \\ 2\mathsf{D}^2 - \mathsf{D} & 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \xi_1'(t) \\ \xi_2'(t) \end{bmatrix}\right)^{\mathsf{T}}$$

is a linear map of the left solution space of the homogeneous differential equations

$$[\xi_1'(t), \xi_2'(t)]\begin{bmatrix} 1 & \mathsf{D}^2 \\ 0 & 1 \end{bmatrix} = \mathbf{0}^{\mathsf{T}},$$

while

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & \mathsf{D}^2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$$

is not a linear mapping of the right solution set of

$$\begin{bmatrix} 1 & \mathsf{D} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \mathbf{0},$$

because the McMillan-degree condition for the compound matrix $[M(s), T_2(s)]$ is satisfied (see (17)) but the McMillan-degree condition for the second compound matrix (18) is not. □

An explanation can be given, based on Theorem 2, of why any map between the solution set of any homogeneous system of ordinary differential equations and the solution set of a homogeneous system in state-space form is always constant [15].

THEOREM 3 Consider the homogeneous systems

$$(S_1) \quad E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t), \qquad (S_2) \quad T(\mathsf{D})\boldsymbol{\xi}(\mathrm{t}) = \mathbf{0}, \tag{45}$$

where $T(s), sE - A \in \mathcal{P}(p, m)$. If

$$\boldsymbol{\xi}(t) = N(\mathsf{D})\boldsymbol{x}(t) \tag{46}$$

is a map between $\boldsymbol{\xi}(t)$ and $\boldsymbol{x}(t)$, where $N(s) = N_q s^q + \cdots + N_1 s + N_0$, then $N(s)$ is constant.

Proof. By theorem 2, a necessary and sufficient condition for (46) to be a map is

$$\delta_{\mathrm{M}}\left(\begin{bmatrix} sE - A \\ N(s) \end{bmatrix}\right) = \delta_{\mathrm{M}}(sE - A),$$

or equivalently [5]

$$\operatorname{rank}_{\mathbb{R}}\begin{bmatrix} E & O & \dots & O \\ N_q & O & \dots & O \\ N_{q-1} & N_q & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ N_1 & N_2 & \dots & N_q \end{bmatrix} = \operatorname{rank}_{\mathbb{R}}[E],$$

i.e. $N_1 = HE$, for some constant matrix H, and $N_i = O \ (i = 2, \dots, q)$. Thus

$$\boldsymbol{\xi}(t) = (N_0 + N_1\mathsf{D})\boldsymbol{x}(t) = (N_0 + HE\,\mathsf{D})\boldsymbol{x}(t) = (N_0 + HA)\boldsymbol{x}(t) \quad (\text{ by (45)}),$$

and so the theorem has been proved. □

EXAMPLE 5 Consider the homogeneous system

$$(S_1) \quad \begin{bmatrix} 1 & \mathsf{D}^2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = O$$

and the system in state-space form

$$(S_2) \quad \begin{bmatrix} 1 & \mathsf{D} & 0 \\ 0 & 1 & \mathsf{D} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \mathbf{0}.$$

By forming the McMillan-degree conditions, it can be seen that

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

is a constant map between the solution of S_2 and S_1. □

4. Conclusions

Some partial explanations of the role of the McMillan-degree conditions which appear in the transformation of full equivalence [3] have been presented. A connection between this condition and the recent work by Zhang [13] has also been given. These explanations however appear to highlight the role that the conditions play in the proof of invariance of the zero structure of two matrices related by a transformation of full equivalence. In this sense, the explanations serve to underline the conditions as a device rather than a fundamental requirement of the theory. However a complete explanation emerges in terms of maps between the solution spaces of the associated dynamical systems, and it is thus apparent that the McMillan-degree conditions are of fundamental importance in the study of equivalence of linear systems.

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