A NOTE ON THE ACTION OF CONSTANT PSEUDOSTATE FEEDBACK ON THE INTERNAL PROPERNESS OF AN ARMA MODEL*

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Abstract In this note we study the effect of constant pseudostate feedback on the internal properness of a linear multivariable system, described by an ARMA model. It is shown that the existence of a constant pseudostate feedback control law which makes the closed loop system internally proper is equivalent to the absence of decoupling zeros at infinity of the open loop system, a well known result from the theory of descriptor systems.

1. Introduction

We consider systems described by ARMA models having the form

$$A(\rho)\xi(t) = B(\rho)u(t) \tag{1.1}$$

where $A(\rho) = \sum_{i=1}^{q} A_i\rho^i \in R[\rho]^{r\times r}$, $B(\rho) = \sum_{i=1}^{p} B_i\rho^i \in R[\rho]^{r\times m}$, $\xi(t)$ is the 'pseudostate' vector and $u(t)$ is the input vector, ρ stands either for the differential operator $\frac{d}{dt}$ in the continuous time or the time advance operator $\rho\xi(t) = \xi(t+1)$ in the discrete time case. We assume that (1.1) is *regular*, i.e. $\det A(\rho) \neq 0$ for almost every ρ, which guarantees the uniqueness of the solution given the initial conditions and the input. The term 'pseudostate' is justified by the fact that $\xi(t)$ can be considered as the vector of internal or 'latent' variables of the system (see [13]).

Consider now (1.1) together with the pseudostate feedback control law

$$u(t) = K\xi(t) + v(t) \tag{1.2}$$

where $K \in R^{r\times m}$ and $v(t)$ is a new input. The objective of this note is to derive a necessary and sufficient condition, under which the closed loop system described by (1.1) and (1.2) is *internally proper*.

The significance of *internal properness* of a continuous-time, linear system arises from the fact that its absence gives rise to impulsive behavior, either because of inconsistent initial conditions or due to the presence of discontinuous input signals. In general, impulsive behavior is an undesirable feature for a

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system. Furthermore, many polynomial design techniques require systems that are already internally proper (see for example [2], [10]-Ch.7). In the discrete-time case the lack of internal properness reflects non-casual behavior of the underlying system. This type of behavior occurs in many discrete-time economical or social models (e.g. the Leontief model, see [7]) and internal properness of such a system simply translates to *causality*. In what follows we shall only refer to the continuous case since the results are identical for both cases.

The action of constant pseudostate (output) feedback K on the (external) properness of the closed loop transfer function matrix of (1.1) has been studied in [1], where it is shown that generically there exists a K which makes the closed loop transfer function matrix proper. Furthermore in [9] conditions for K giving rise to a non-proper closed loop transfer function have been derived.

The theory of descriptor systems has developed conditions under which a generalized state space system can be made internally proper by the use of state (descriptor) feedback. Such results can be found in [3], [4], [5], [7] where a geometric characterization of impulse controllability and its role on the existence of a state feedback that makes the system 'impulse free', is given. For instance the elimination of impulsive behavior is a necessary step in order to apply LQ control techniques on a descriptor system in [4]. The related problem of controllability at infinity is discussed in [6], [8] and [12]. In particular in [6] it is shown that controllability (finite and infinite) is one of the necessary conditions for arbitrary pole placement via generalized state feedback.

In what follows R, C denote the field of real and complex numbers respectively, $R[\rho]$ the ring of polynomials with coefficients in R, $R(\rho)$ the field of real rational functions and $R_{pr}(\rho)$ the ring of (real) proper rational functions. The superscripts in the above symbols denote sets of matrices or vectors having their elements in the corresponding ring or field.

2. Main results

We give first a definition of the McMillan degree of a general rational matrix (see e.g. [10]):

Definition 1. *The McMillan degree of a rational matrix $A(\rho) \in R(\rho)^{p\times m}$ is defined as the total number of poles in $C \cup \{\infty\}$ of $A(\rho)$, i.e.*

$$\delta_M(A(\rho)) := \{\#\text{ of poles in } C \cup \infty\}$$

We notice that in case $A(\rho)$ is a polynomial matrix, its McMillan degree equals to the total number of poles at $\rho = \infty$, since there are no finite poles.

The definition of internal properness given bellow is a direct consequence of the definitions given in [2]-pp.114 where internally proper systems are termed 'well-formed' or in [10]-pp.240. The abovementioned definitions are a bit more general since they involve an output vector as well. However, in our case the pseudostate vector can be considered as the output of the system.

Definition 2. *The system (1.1), is said to be internally proper iff*

(i) for every initial value $\xi(0-)$ *and its derivatives* $\xi^{(i)}(0-)$, $i = 1, 2, ..., q-1$

(ii) for every 'impulse free' input $u(t)$ *with* $u^{(i)}(0-) = 0$, $i = 0, 1, 2, ...$

the pseudostate $\xi(t)$ *is 'impulse free', i.e. it does not contain Dirac impulses* $\delta(t)$ *and its derivatives* $\delta^{(i)}(t)$.

We give now some results regarding the internal properness of (1.1).

Lemma 3. *The following statements are equivalent*

(i) The system described by (1.1) is internally proper.

(ii) $A^{-1}(\rho) \in R_{pr}^{r\times r}(\rho)$, $A^{-1}(\rho)B(\rho) \in R_{pr}^{r\times m}(\rho)$.

(iii) The polynomial matrix $R(\rho) := \begin{bmatrix} A(\rho) & B(\rho) \\ 0 & I_m \end{bmatrix}$ *has no zeros at* $\rho = \infty$.

(iv) $\deg |A(\rho)| = \delta_M \begin{bmatrix} A(\rho), & B(\rho) \end{bmatrix}$.

Proof. The equivalence of statements (i),(ii) and (iv) is a direct consequence of theorem 52, pp. 115 in [2], while the equivalence of (ii) and (iii) is established after some trivial manipulations in theorem 4.90, pp. 240 in [10]. ■

Condition (ii) in the above lemma states that the properness of the transfer function matrix $A^{-1}(\rho)B(\rho)$ cannot guarantee impulse free behavior for the system. This is due to fact that even if the input-pseudostate transfer function is proper, there might still be impulsive behavior in the free pseudostate response due to appropriate initial conditions $\xi^{(i)}(0-), i = 0, 1, ..., q-1$.

Remark 1. *The above lemma can be considered as a generalization of a well known result from the theory of descriptor systems. If we set* $A(\rho) = \rho E - F$ *and* $B(\rho) = G$ *then condition (iv) of lemma 3 becomes*

The system described by $(\rho E - F)\, x(t) = G u(t)$ is internally proper $\Leftrightarrow$

$$(\rho E - F)^{-1} \in R_{pr}^{r\times r}(\rho) \Leftrightarrow \deg |\rho E - F| = \tag{2.1}$$

$$\delta_M([\ \rho E - F, \quad G\ ]) = \delta_M(\rho E - F) = rank E \tag{2.2}$$

This result occurs in several studies (see for example [8], [7]).

The following definition can be considered as a special case of the definition of input decoupling zeros at infinity of a general polynomial matrix description of a system, which appears in [11].

Definition 4. *The decoupling zeros at* $\rho = \infty$ *of (1.1) are the zeros at* $\rho = \infty$ *of* $\begin{bmatrix} A(\rho), & B(\rho) \end{bmatrix}$.

Consider now (1.1) together with the following pseudostate feedback

$$u(t) = K\xi(t) + v(t) \tag{2.3}$$

where $K \in R^{r\times m}$ and $v(t)$ is a new input. Then the closed loop system is described by

$$[A(\rho) + B(\rho)K]\,\xi(t) = B(\rho)v(t) \tag{2.4}$$

Definition 5. *The pseudostate feedback law K is called admissible iff (2.4) is regular i.e. iff* $\det[A(\rho) + B(\rho)K] \neq 0$ *for almost every ρ.*

Consider now a left coprime at $\rho = \infty$ proper rational matrix fractional representation of $\left[\begin{array}{cc} A(\rho), & B(\rho)\end{array}\right]$, i.e. let

$$\left[\begin{array}{cc} A(\rho), & B(\rho)\end{array}\right] = D(\rho)^{-1}\left[\begin{array}{cc} N_A(\rho), & N_B(\rho)\end{array}\right] \tag{2.5}$$

where $D(\rho) \in R_{pr}^{r\times r}(\rho)$, $N_A(\rho) \in R_{pr}^{r\times r}(\rho)$,$N_B(\rho) \in R_{pr}^{r\times m}(\rho)$ and $rank\left[\begin{array}{ccc} D(\infty), & N_A(\infty), & N_B(\infty)\end{array}\right] = r$. Then we have

Fact 1. The zeros at $\rho = \infty$ of $\left[\begin{array}{cc} A(\rho), & B(\rho)\end{array}\right]$ are the zeros at $\rho = \infty$ of $\left[\begin{array}{cc} N_A(\rho), & N_B(\rho)\end{array}\right]$ [10].

Consider also the polynomial matrix

$$R_K(\rho) := \left[\begin{array}{cc} A(\rho)+B(\rho)K & B(\rho) \\ 0 & I_m\end{array}\right] = \left[\begin{array}{cc} D(\rho) & 0 \\ 0 & I_m\end{array}\right]^{-1}\left[\begin{array}{cc} N_A(\rho) & N_B(\rho) \\ -K & I_m\end{array}\right]\left[\begin{array}{cc} I_r & 0 \\ K & I_m\end{array}\right] \tag{2.6}$$

It is easy to see that $rank\left[\begin{array}{cccc} D(\infty) & 0 & N_A(\infty) & N_B(\infty) \\ 0 & I & -K & I_m\end{array}\right] = r + m$ so that (2.6) is a left coprime at $\rho = \infty$ proper rational matrix fractional representation of $R_K(\rho)$ and therefore we have

Fact 2. The zeros of $R_K(\rho)$ at $\rho = \infty$ are the zeros at $\rho = \infty$ of $N_K(\rho) := \left[\begin{array}{cc} N_A(\rho) & N_B(\rho) \\ -K & I_m\end{array}\right]$.

We now state our main result.

Theorem 6. *The following statements are equivalent.*

(i) There exists an admissible pseudostate feedback as in (2.3) such that the closed loop system (2.4) is internally proper

(ii) $N_K(\rho) := \left[\begin{array}{cc} N_A(\rho) & N_B(\rho) \\ -K & I_m\end{array}\right]$ *has no zeros at $\rho = \infty$*

(iii) The system (1.1) has no decoupling zeros at $\rho = \infty$.

Proof. (i) $\Rightarrow$(ii) Assume that there exists a K as in (i). Then from Lemma 3 $R_K(\rho)$ has no zeros at $\rho = \infty$ which in view of Fact 2 implies (ii).

(ii)$\Rightarrow$(iii) $N_K(\rho)$ has no zeros at $\rho = \infty$ implies $rank N_K(\infty) = r+m$ which in turn implies that $rank[\begin{array}{cc} N_A(\infty), & N_B(\infty)\end{array}] = r$ which due to Fact 1 implies that $\left[\begin{array}{cc} A(\rho), & B(\rho)\end{array}\right]$has no zeros at $\rho = \infty$ or from Definition 4 implies (iii).

(iii)⇒(i) Assume that he system (1.1) has no decoupling zeros at $\rho = \infty$ or equivalently that $\left[\begin{array}{cc} A(\rho), & B(\rho) \end{array}\right]$ has no zeros at $\rho = \infty$. From Fact 1 this implies that $\left[\begin{array}{cc} N_A(\rho), & N_B(\rho) \end{array}\right]$ has no zeros at $\rho = \infty$ or that $rank \left[\begin{array}{cc} N_A(\infty), & N_B(\infty) \end{array}\right] = r$ or equivalently that $rank \left[\begin{array}{cc} sI_r - N_A(\infty), & N_B(\infty) \end{array}\right] = r$, for $s = 0$. In other words the pair of matrices $(sI_r - N_A(\infty), N_B(\infty))$ has no input decoupling zeros at $s = 0$. This guarantees the existence of an appropriate (state) feedback K which assigns the (possible) zero eigenvalues of $N_A(\infty)$ to any arbitrary position in the $C-$ plane, i.e. such that $\det [N_A(\infty) + N_B(\infty)K] \neq 0$. Now it is easy to see that $\det N_K(\infty) = \det [N_A(\infty) + N_B(\infty)K] \neq 0$. Thus there exists a K such that $rank N_K(\infty) = r + m$, i.e. $N_K(\rho) \in R_{pr}^{(r+m)\times(r+m)}(s)$ is biproper which implies:

- (a) $\det N_K(\rho) \neq 0$ for almost every ρ which implies that $\det R_K(\rho) \neq 0$ for almost every ρ which, from (2.6), implies that $\det [A(\rho) + B(\rho)K] \neq 0$ for almost every ρ i.e. the closed loop system (2.4) is *regular* or equivalently that the pseudostate feedback law K is *admissible* and
- (b) $R_K(\rho)$ has no zeros at $\rho = \infty$, which from Lemma 3 (iii) implies that he system (1.1) is *internally proper*. ■

The proof of the above theorem suggests a way to obtain K

- Calculate a coprime at $\rho = \infty$ proper rational matrix fractional representation of $[A(\rho), B(\rho)]$ as in (2.5),
- Find a K such that $\det [N_A(\infty) + N_B(\infty)K] \neq 0$.

We illustrate this result via the following

Example 7. *Consider the system described by*

$$\underbrace{\left[\begin{array}{cc} \rho^2 - 1 & \rho + 1 \\ \rho - 2 & 1 \end{array}\right]}_{A(\rho)} \xi(t) = \underbrace{\left[\begin{array}{c} \rho - 1 \\ \rho^2 \end{array}\right]}_{B(\rho)} u(t) \tag{2.7}$$

Since

$$A(\rho)^{-1} = \left[\begin{array}{cc} \frac{1}{\rho+1} & -1 \\ -\frac{\rho-2}{\rho+1} & \rho - 1 \end{array}\right] \notin R_{pr}^{2\times 2}(\rho)$$

$$A(\rho)^{-1}B(\rho) = \left[\begin{array}{c} -\frac{\left(\rho^3+\rho^2-\rho+1\right)}{\rho+1} \\ \frac{(\rho-1)\left(\rho^3+\rho^2-\rho+2\right)}{\rho+1} \end{array}\right] \notin R_{pr}^{2\times 1}(\rho)$$

the system is not internally proper. The Smith-McMillan form of $[A(\rho), B(\rho)]$ at $\rho = \infty$ is

$$S^{\infty}_{[A(\rho),B(\rho)]} = \left[\begin{array}{ccc} \rho^2 & 0 & 0 \\ 0 & \rho^2 & 0 \end{array}\right] \tag{2.8}$$

Obviously $[A(\rho), B(\rho)]$ has no zeros at $\rho = \infty$, i.e. (2.7) has no decoupling zeros at $\rho = \infty$, which implies the existence of a constant feedback $K = [k_1, k_2]$ such the closed loop system (2.4) is internally proper. We calculate a coprime at $\rho = \infty$ proper rational matrix fractional representation of $[A(\rho), B(\rho)]$ with

$$D(\rho) = \begin{bmatrix} 1/\rho^2 & 0 \\ 0 & 1/\rho^2 \end{bmatrix}, [N_A(\rho), N_B(\rho)] = \begin{bmatrix} 1 - \frac{1}{\rho^2} & \frac{(\rho+1)}{\rho^2} & \frac{\rho-1}{\rho^2} \\ \frac{\rho-2}{\rho^2} & \frac{1}{\rho^2} & 1 \end{bmatrix} \tag{2.9}$$

Now it is enough to find a K such that

$$\det\left[N_A(\infty) + N_B(\infty)K\right] = \det \begin{bmatrix} 1 & 0 \\ k_1 & k_2 \end{bmatrix} \neq 0 \tag{2.10}$$

Thus any K with $k_2 \neq 0$ can make the closed loop system internally proper. For simplicity choose $k_1 = 0$ and $k_2 = 1$. With this feedback the closed loop system (2.4) is given by

$$\begin{bmatrix} \rho^2 - 1 & 2\rho \\ \rho - 2 & 1 + \rho^2 \end{bmatrix} \xi(t) = \begin{bmatrix} \rho - 1 \\ \rho^2 \end{bmatrix} v(t) \tag{2.11}$$

and thus from the facts that

$$\begin{aligned} \left[A(\rho) + B(\rho)K\right]^{-1} &= \begin{bmatrix} \frac{1+\rho^2}{\rho^4-1-2\rho^2+4\rho} & \frac{-2\rho}{\rho^4-1-2\rho^2+4\rho} \\ -\frac{\rho-2}{\rho^4-1-2\rho^2+4\rho} & \frac{\rho^2-1}{\rho^4-1-2\rho^2+4\rho} \end{bmatrix} \in R_{pr}^{2\times 2}(\rho) \\ \left[A(\rho) + B(\rho)K\right]^{-1} B(\rho) &= \begin{bmatrix} -\frac{-\rho+1+\rho^3+\rho^2}{\rho^4-1-2\rho^2+4\rho} \\ \frac{-2\rho^2+3\rho-2+\rho^4}{\rho^4-1-2\rho^2+4\rho} \end{bmatrix} \in R_{pr}^{2\times 1}(\rho) \end{aligned}$$

(2.11) is internally proper. ■

3. Conclusions

In this note we have proposed a method for the elimination of the undesirable impulsive behavior of a linear system described by an ARMA representation, using constant pseudostate feedback. It has been shown that a necessary and sufficient condition for the existence of such a feedback, is the absence of decoupling zeros at infinity of the open loop system. This condition appears to be a generalization of known results from the theory of descriptor systems and particularly the ones regarding the impulse controllability of a generalized state space system.

References

[1] R.W. Scott, B.D.O. Anderson, 'Comments on Conditions for a feedback Transfer Matrix to be Proper', IEEE Trans. Autom. Control, August 1976, pp. 632-634.

[2] F.M. Callier, C.A. Desoer, 'Multivariable Feedback Systems', Springer - Verlag, New York, 1982.

[3] D. Cobb, 'Descriptor variable and Generalized Singularly Perturbed Systems: A Geometric Approach', Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1980.

[4] D. Cobb, 'Descriptor Variable Systems and Optimal State regulation', IEEE Trans. Autom. Control, vol. AC-28, No 5, May 1983, pp. 601-611.

[5] D. Cobb, 'Controllabilty, Observabillty and Duality in Singular Systems', IEEE Trans. Autom. Control, vol. AC-29, No 12, Dec 1983, pp. 1076-1082.

[6] V. Kucera, P. Zagalak, 'Fundamental Theorem of State Feedback for Singular Systems', Automatica, vol. 24, No 5, 1988, pp. 653-658.

[7] F.L. Lewis, 'A Survey of Linear Singular Systems', Circuits, Systems & Signal Processing, vol. 5, No 1, 1986.

[8] T. Yamada, D.G. Luenberger, 'Generic Controllability Theorems for Descriptor Systems', IEEE Trans. Autom. Control, vol. AC-30, No 2, Feb 1985, pp. 144-152.

[9] A.C. Pugh, 'The Occurrence of Non-Properness in Closed-Loop Systems and some implications', Multivariable Control - New Concepts and Tools, ed. S.G. Tzefestas, D. Reidel Publishing Company, 1984, pp. 43-63.

[10] A.I.G. Vardulakis, 'Linear Multivariable Control - Algebraic Analysis and Synthesis Methods', John Wiley & Sons, New York, 1991.

[11] G.C. Verghese, 'Infinite frequency Behaviour in Dynamical Systems', Ph.D. Dissertation, Dept. of Electrical Engineering, Standford University, 1978.

[12] G.C. Verghese, B.C. Levy, T. Kailath, 'A Generalized State-Space for Singular Systems', IEEE Trans. Autom. Control, vol. AC-26, No 4, Aug 1981, pp. 811-831.

[13] J.C. Willems, 'Paradigms and Puzzles in the Theory of Dynamical Systems', IEEE Trans. Autom. Control, Vol. 36, No 3, March 1991, pp. 259-294.