On the solution and impulsive behavior of polynomial matrix descriptions of free linear multivariable systems

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Abstract In this note we examine the solution and the impulsive behavior of autonomous linear multivariable systems whose pseudo-state $\beta(t)$ obeys a linear matrix differential equation $A(\rho)\beta(t) = 0$ where $A(\rho)$ is a polynomial matrix in the differential operator $\rho := \frac{d}{dt}$. We thus generalize to the general polynomial matrix case some results obtained in [2][3] which regard the impulsive behavior of the generalized state vector $x(t)$ of input free *generalized state space* systems.

1. Introduction

Consider a free system whose dynamics are described by the linear homogenous matrix differential equation

$$A(\rho)\beta(t) = 0 \quad t \geq 0 \tag{1.1}$$

where

$$A(\rho) = A_q\rho^q + A_{q-1}\rho^{q-1} + \ldots + A_1\rho + A_0 \in \mathbf{R}[\rho]^{r\times r} \tag{1.2}$$

is a polynomial matrix in $\rho = \frac{d}{dt}$, $A_i \in \mathbf{R}^{r\times r}, i = 0, 1, 2, \ldots, q > 0$ $rank_{\mathbf{R}(\rho)}A(\rho) = r$ and $\beta(t) : [0, \infty) \to \mathbf{R}^r$ is what is known as the *pseudo-state* of the system.

In this note we firstly review the fact that if $A(s)^{-1}$ is a non-proper rational matrix then depending on the choice of the *initial values* $\beta(0-)$, $\beta^{(1)}(0-), \ldots, \beta^{(q-1)}(0-)$, (where $\beta^{(i)}(t) := \frac{d^i\beta(t)}{dt^i}$) the solution $\beta(t)$ of (1.1) might exhibit an '*impulsive behavior*' at $t = 0$ which consists of a combination of the Dirac impulse $\delta(t)$ and its $(\widehat{q}_r - 1)$-*th* order distributional derivatives (where $\widehat{q}_r$ is the maximum order of the zero at $s = \infty$ of $A(s)$, see below). Due to the fact that $A(s)^{-1}$ is a non-proper rational matrix if and only if $A(s)$ has zeros at $s = \infty$, the impulsive behavior of $\beta(t)$ at $t = 0$ for appropriate initial values can be seen as being associated to the *zero structure at* $s = \infty$ of $A(s)$, i.e. due to the fact that the natural modes of (1.1), defined as values of s where $A(s)$ loses rank, include also the point at $s = \infty$. Based on these facts and assuming that $A(s)$ has zeros at $s = \infty$ we then characterize the set of initial values $\beta^{(i)}(0-), i = 0, 1, 2, ..., q-1$ that are such so that $\beta(t)$ has no impulsive behavior at $t = 0$. Furthermore we characterize the set of initial values that are such that not only $\beta(t)$ but also its derivatives $\beta^{(i)}(t)$ up to a certain order $i = 1, 2, ..., j \leq q-1$ are continuous at $t = 0$ so that $\beta^{(i)}(0-) = \beta^{(i)}(0+)$, $i = 0, 1, 2, \ldots q-1$. We then examine conditions that

$A(s)^{-1}$ has to satisfy so that $\beta(t)$ has not impulsive behavior at $t=0$ *for every set of initial values* $\beta^{(i)}(0-)$, $i=0,1,2,...,q-1$. A necessary and sufficient condition for the continuity of $\beta(t)$ at $t=0$ and for every set of initial values $\beta^{(i)}(0-)$, $i=0,1,2,...,q-1$ in terms of coefficient in the Laurent expansion of $A(s)^{-1}$ at $s=\infty$ is given in Proposition 1. This result is then generalized by giving necessary and sufficient conditions for the continuity of $\beta(t)$ *and of all its derivatives* $\beta^{(i)}(t)$ up to order $j\leq q-1$ and for every set of initial values at $t=0-$: $\beta^{(i)}(0-)$, $i=0,1,2,\ldots,q-1$. The results about the continuity of $\beta(t)$ and its derivatives at $t=0$ preseneted here, are comparable to those in [6][7] where the notions of consistency and weak constintency have been introduced.

We thus generalize to the general polynomial matrix case some results obtained in [2][3] regarding the response and the impulsive behavior of the generalized state vector $x(t):(0-,\infty)\rightarrow\mathbf{R}^n$ of input free *generalized state space* systems i.e. linear systems whose state vector $x(t)$ is governed by the generalized state space equation

$$E\dot{x}(t)=Ax(t)\quad t\geq 0 \tag{1.3}$$

where $E\in\mathbf{R}^{n\times n}$, $A\in\mathbf{R}^{n\times n}$ and $rank_{\mathbf{R}}E\leq n$ and which are associated with finite and infinite zero structure of the *matrix pencil* $sE-A$.

2. Background

In this section we review a number of results required in the sequel. This background comes mainly from [1]. In the following $\mathbf{R}$ denotes the field of reals, $\mathbf{R}[s]$ the ring of polynomials, $\mathbf{R}(s)$ the field of rational functions and $\mathbf{R}_{pr}(s)$ the ring of *proper* rational functions all in the indeterminate s and with coefficients in $\mathbf{R}$. If k is a set then $k^{p\times m}$ denotes the set of $p\times m$ matrices with elements in k. If $T(s)\in\mathbf{R}(s)^{p\times m}$, $\delta_M(T(s))$ denotes the *McMillan degree* of the $T(s)$ i.e., its total number of poles (finite and at $s=\infty$ and multiplicities accounted for).

Consider a polynomial matrix

$$A(s)=A_q s^q+A_{q-1}s^{q-1}+\ldots+A_0\in\ \mathbf{R}[s]^{r\times r} \tag{2.1}$$

where $A_i\in\mathbf{R}^{r\times r}$ $i=0,1,\ldots,q$, $A_q\neq 0$ with $rank_{\mathbf{R}(s)}A(s)=r, q\geq 1$and let

$$S^{\infty}_{A(s)}=\mathrm{diag}\left[\overbrace{\underbrace{s^{q_1},s^{q_2},\cdots,s^{q_k}}_{k},I_{v-k}}^{v},\overbrace{\frac{1}{s^{\widehat{q}_{v+1}}},\cdots,\frac{1}{s^{\widehat{q}_r}}}^{r-v}\right] \tag{2.2}$$

be the Smith-McMillan form of $A(s)$ at $s=\infty$ [1] where $0\leq k\leq v\leq r$, and $q_1\geq q_2\geq\ldots\geq q_k>0=q_{k+1}=\cdots=q_v$, $\widehat{q}_r\geq\widehat{q}_{r-1}\geq\cdots\geq\widehat{q}_{v+1}>0$ are respectively the orders of the *poles* and the *zeros* at $s=\infty$ of $A(s)$. Then the following facts hold true:

Fact 1.[1]

$$q_1=q \tag{2.3}$$

Fact 2. The Laurent series expansion at $s=\infty$ of the rational matrix $A(s)^{-1}\in\mathbf{R}(s)^{r\times r}$ has the form [1]

$$A(s)^{-1}=H_{\widehat{q}_r}s^{\widehat{q}_r}+H_{\widehat{q}_r-1}s^{\widehat{q}_r-1}+\cdots+H_1s+H_0+H_{-1}s^{-1}+H_{-2}s^{-2}+\cdots$$
$$=H_{pol}(s)+H_{sp}(s) \tag{2.4}$$

where $H_{pol}(s)=H_{\widehat{q}_r}s^{\widehat{q}_r}+H_{\widehat{q}_r-1}s^{\widehat{q}_r-1}+\cdots+H_1s+H_0\in\mathbf{R}[s]^{r\times r}$, $H_i\in\mathbf{R}^{r\times r}, i=0,1,\ldots,\widehat{q}_r$, $H_{\widehat{q}_r}\neq 0$ and $H_{sp}(s)=H_{-1}s^{-1}+H_{-2}s^{-2}+\cdots\in\mathbf{R}_{pr}(s)^{r\times r}$ is strictly proper. From the fact that $A(s)^{-1}A(s)=I_r$, it is obvious that the terms H_i in (2.4) satisfy the following identities

$$H_{i-q_1}A_{q_1}+H_{i-q_1+1}A_{q_1-1}+...+H_iA_0 \tag{2.5}$$

$$=A_{q_1}H_{i-q_1}+A_{q_1-1}H_{i-q_1+1}+...+A_0H_i=\delta_iI_r\ \ ,\forall i \tag{2.6}$$

where $\delta_i=0$ for $i\neq 0$ and $\delta_0=1$ (terms H_i, with $i>\hat{q}_r$ are zero).

If we consider the matrix pair $[I_r,A(s)]$ which is trivially right coprime then from the polynomial matrix (right) division of I_r by $A(s)$ [4] there exist $Q(s),R(s)\in\mathbf{R}[s]^{r\times r}$ such that

$$I_r=Q(s)A(s)+R(s) \tag{2.7}$$

or

$$A(s)^{-1}=Q(s)+R(s)A(s)^{-1}=H_{pol}(s)+H_{sp}(s) \tag{2.8}$$

where $H_{pol}(s):=Q(s)$ and $H_{sp}(s):=R(s)A(s)^{-1}$. Eq. (2.7) can be written as

$$\begin{bmatrix} I_r \\ A(s)\end{bmatrix}=\begin{bmatrix} I_r & Q(s) \\ 0_{r,r} & I_r\end{bmatrix}\begin{bmatrix} R(s) \\ A(s)\end{bmatrix} \tag{2.9}$$

which implies that the pair $[R(s),A(s)]$ is also right coprime and thus we have

Fact 3. $\delta_M\left(H_{sp}(s)\right)=\deg|A(s)|=:n.$

Fact 4. The Smith-McMillan form of $H_{pol}(s)$ at $s=\infty$ has the form [1]

$$S^{\infty}_{H_{pol}(s)}=\text{diag}\left[\overset{\xleftrightarrow{\quad d\quad}}{\underset{\xleftrightarrow{r-v}}{s^{\widehat{q}_r},s^{\widehat{q}_r-1},\cdots,s^{\widehat{q}_v+1}}},I_{d-(r-v)},\frac{1}{s^{\widetilde{q}_{d+1}}},\cdots\frac{1}{s^{\widetilde{q}_\sigma}},0_{r-\sigma,r-\sigma}\right] \tag{2.10}$$

where $1\leq(r-v)\leq d\leq\sigma=rank_{\mathbf{R}(s)}H_{pol}(s)$ and $\widetilde{q}_\sigma\geq\widetilde{q}_{\sigma-1}\geq\cdots\geq\widetilde{q}_{d+1}>0$ are the orders of the zeros at $s=\infty$ of $H_{pol}(s)$ i.e. the *pole structure at* $s=\infty$ of $A(s)^{-1}$ (which is the *zero structure at* $s=\infty$ of $A(s)$) coincides with the *pole structure at* $s=\infty$ of its polynomial part $H_{pol}(s)$.

Finally let $C_\infty\in\mathbf{R}^{r\times\mu}$, $J_\infty\in\mathbf{R}^{\mu\times\mu}$, $B_\infty\in\mathbf{R}^{\mu\times r}$, be an *irreducible at* $s=\infty$ [2][3] *generalized state space realization* of $H_{pol}(s)$ i.e. let $\frac{1}{w}H_{pol}\left(\frac{1}{w}\right)=C_\infty\left(wI_\mu-J_\infty\right)^{-1}B_\infty$ so that with $\frac{1}{w}=s$:

$$H_{pol}(s)=C_\infty\left(I_\mu-sJ_\infty\right)^{-1}B_\infty \tag{2.11}$$

where [1]

$$\mu=\delta_M\left[\frac{1}{w}H_{pol}\left(\frac{1}{w}\right)\right]=\sum_{i=v+1}^{r}\left(\widehat{q}_i+1\right)+\left[d-(r-v)\right] \tag{2.12}$$

$$J_\infty=block\ diag\left[0_{d-(r-v),d-(r-v)},\widetilde{J}_{\infty v+1},\widetilde{J}_{\infty v+2},\ldots,\widetilde{J}_{\infty r}\right]\in\mathbf{R}^{\mu\times\mu} \tag{2.13}$$

$$\widetilde{J}_{\infty i}=\left[\begin{array}{ccccc}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{array}\right]\in\mathbf{R}^{\left(\widehat{q}_i+1\right)\times\left(\widehat{q}_i+1\right)}\qquad i=v+1,\ldots,r \tag{2.14}$$

and

$$rank_{\mathbf{R}}\left[\begin{array}{c}C_\infty\\C_\infty J_\infty\\\vdots\\C_\infty J_\infty^{\mu-1}\end{array}\right]=rank_{\mathbf{R}}\left[\begin{array}{cccc}B_\infty&J_\infty B_\infty&\ldots&J_\infty^{\mu-1}\end{array}\right]=\mu \tag{2.15}$$

Let also $C\in\mathbf{R}^{r\times n}$, $J\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times r}$ be a *minimal* state space realization of $H_{sp}(s)$ with J in Jordan normal form i.e. let

$$H_{sp}(s)=C\left(sI_n-J\right)^{-1}B \tag{2.16}$$

Then from (2.4)(2.11) and (2.16) we have

Fact 5.

$$H_i=C_\infty J_\infty^i B_\infty\qquad i=0,1,2,\ldots,\widehat{q}_r \tag{2.17}$$

$$H_{-i}=CJ^{i-1}B\qquad i=1,2,\ldots \tag{2.18}$$

3. Solution of Linear Homogeneous matrix differential equations and impulsive behavior of their solution at $t=0$

Consider the homogeneous matrix differential equation in (1.1)(1.2). In this section we examine the solution of (1.1) for every possible value of $\beta(t)$ and its derivatives at $t=0-$ using the Laplace transform method. Let $\beta(0-),\beta^{(1)}(0-),\ldots,\beta^{(\kappa-1)}(0-)$ be the initial values of the pseudo-state $\beta(t)$ and its derivatives up to order $q-1$ at $t=0-$. As it will be seen in the sequel by allowing $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i=1,2,\ldots$ to have arbitrary values at $t=0-$ we do not guarantee that $\beta(t)$ is continuous at $t=0$ i.e. we might have that $\beta^{(i)}(0-)\neq\beta^{(i)}(0+)$, $i=0,1,2,\ldots$
Considering the L_- Laplace transform $\widehat{\beta}(s)$ of $\beta(t)$: $\widehat{\beta}(s):=L_-\beta(t)=\int_{0-}^{\infty}\beta(t)e^{-st}dt$ and taking the L_- Laplace transform of (1.1) we obtain

$$L_-\left\{A(\rho)\beta(t)\right\}=A(s)\widehat{\beta}(s)-\widehat{\alpha}(s)=0 \tag{3.1}$$

where $A(s)=A_{q_1}s^{q_1}+A_{q_1-1}s^{q_1-1}+\ldots+A_0\in\mathbf{R}[s]^{r\times r}$ and

$$\widehat{\alpha}(s)=\left[s^{q_1-1}I_r,s^{q_1-2}I_r,\ldots,sI_r,I_r\right]$$

$$
\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_2 & A_3 & \dots & A_{q_1} & 0 \\ A_1 & A_2 & \dots & A_{q_1-1} & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-2)}(0-) \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \mathbf{R}[s]^{r\times 1} \tag{3.2}
$$

is the *initial condition* vector associated with the initial values $\beta(0-), \beta^{(1)}(0-), \dots, \beta^{(q-1)}(0-)$. From (3.1) we obtain

$$
\widehat{\beta}(s) = A(s)^{-1}\,\widehat{\alpha}(s) \in \mathbf{R}(s)^{r\times 1} \tag{3.3}
$$

i.e. the L_- Laplace transform $\widehat{\beta}(s)$ of $\beta(t)$ will be in general a possibly non-proper rational vector. Going back to (3.3) and using (2.4) and (3.2) for $\widehat{q}_r \geq q_1$ (and with appropriate changes for $\widehat{q}_r < q_1$) we obtain

$$
\begin{aligned}
\widehat{\beta}(s) &= \left[H_{\widehat{q}_r} s^{\widehat{q}_r} + H_{\widehat{q}_r-1} s^{\widehat{q}_r-1} + \dots + H_1 s + H_0 + H_{-1}\frac{1}{s} + \dots\right]\widehat{\alpha}(s) \\
&= \left[s^{\widehat{q}_r} I_r, s^{\widehat{q}_r-1} I_r, \dots s I_r, I_r, \frac{1}{s} I_r, \dots\right] \begin{bmatrix} H_{\widehat{q}_r} \\ H_{\widehat{q}_r-1} \\ \vdots \\ H_0 \\ H_{-1} \\ \vdots \end{bmatrix} \widehat{\alpha}(s) \\
&= \left[s^{\widehat{q}_r+q_1-1} I_r, s^{\widehat{q}_r+q_1-2} I_r, \dots, s I_r, I_r, \frac{1}{s} I_r, \frac{1}{s^2} I_r \dots\right]
\end{aligned}
$$

$$
\times \underbrace{\left[\begin{array}{ccccc} H_{\widehat{q}_r} & 0 & \dots & 0 & 0 \\ H_{\widehat{q}_r-1} & H_{\widehat{q}_r} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{\widehat{q}_r-(q_1-1)} & H_{\widehat{q}_r-(q_1-2)} & \dots & H_{\widehat{q}_r-1} & H_{\widehat{q}_r} \\ H_{\widehat{q}_r-q_1} & H_{\widehat{q}_r-(q_1-1)} & \dots & H_{\widehat{q}_r-2} & H_{\widehat{q}_r-1} \\ H_{\widehat{q}_r-(q_1+1)} & H_{\widehat{q}_r-q_1} & \dots & H_{\widehat{q}_r-3} & H_{\widehat{q}_r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{-(q_1-2)} & H_{-(q_1-3)} & \dots & H_0 & H_1 \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_{-1} & H_0 \\ - & - & - & - & - \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-2} & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-3} & H_{-2} \\ \vdots & \vdots & & \vdots & \vdots \end{array}\right]}_{rq_1} \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}
$$

$$
= \left[s^{\widehat{q}_r+q_1-1}I_r, s^{\widehat{q}_r+q_1-2}I_r, \ldots, sI_r, I_r\right]
\begin{bmatrix}
H_{\widehat{q}_r} & 0 & \ldots & 0 \\
H_{\widehat{q}_r-1} & H_{\widehat{q}_r} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H_{\widehat{q}_r-(q_1-1)} & H_{\widehat{q}_r-(q_1-2)} & \ldots & H_{\widehat{q}_r} \\
\vdots & \vdots & & \vdots \\
H_{-(q_1-1)} & H_{-(q_1-2)} & \ldots & H_0
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
A_{q_1} & 0 & \ldots & 0 \\
A_{q_1-1} & A_{q_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \ldots & A_{q_1}
\end{bmatrix}
\begin{bmatrix}
\beta(0-) \\
\beta^{(1)}(0-) \\
\vdots \\
\beta^{(q_1-1)}(0-)
\end{bmatrix}
$$

$$
+ \left[\frac{1}{s}I_r, \frac{1}{s^2}I_r \ldots\right]
\begin{bmatrix}
H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\
H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\
\vdots & \vdots & & \vdots
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
A_{q_1} & 0 & \ldots & 0 \\
A_{q_1-1} & A_{q_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \ldots & A_{q_1}
\end{bmatrix}
\begin{bmatrix}
\beta(0-) \\
\beta^{(1)}(0-) \\
\vdots \\
\beta^{(q_1-1)}(0-)
\end{bmatrix}
$$

$$
= \widehat{\beta}_{pol}(s) + \widehat{\beta}_{sp}(s) \tag{3.4}
$$

where $\widehat{\beta}_{pol}(s) \in \mathbf{R}[s]^{r\times 1}$ is the polynomial part and $\widehat{\beta}_{sp}(s) \in \mathbf{R}_{sp}(s)^{r\times 1}$ is the strictly proper part of $\widehat{\beta}(s)$.
Now from (2.5) we obtain the relation

$$
\begin{bmatrix}
H_{\widehat{q}_r} & 0 & \ldots & 0 \\
H_{\widehat{q}_r-1} & H_{\widehat{q}_r} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H_{\widehat{q}_r-(q_1-1)} & H_{\widehat{q}_r-(q_1-2)} & \ldots & H_{\widehat{q}_r} \\
- & - & - & - \\
H_{\widehat{q}_r-q_1} & H_{\widehat{q}_r-(q_1-1)} & \ldots & H_{\widehat{q}_r-1} \\
H_{\widehat{q}_r-(q_1+1)} & H_{\widehat{q}_r-q_1} & \ldots & H_{\widehat{q}_r-2} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
H_{-(q_1-2)} & H_{-(q_1-3)} & \ldots & H_1 \\
H_{-(q_1-1)} & H_{-(q_1-2)} & \ldots & H_0
\end{bmatrix}
\begin{bmatrix}
A_{q_1} & 0 & \ldots & 0 \\
A_{q_1-1} & A_{q_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \ldots & A_{q_1}
\end{bmatrix}
$$

$$= (-1)\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ - & - & - & - \\ H_{\widehat{q_r}} & 0 & \dots & 0 \\ H_{\widehat{q_r}-1} & H_{\widehat{q_r}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\widehat{q_r}-(q_1-1)} & H_{\widehat{q_r}-(q_1-2)} & \dots & H_{\widehat{q_r}} \\ \vdots & \vdots & & \vdots \\ H_2 & H_3 & \dots & H_{q_1+1} \\ H_1 & H_2 & \dots & H_{q_1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix} \quad (3.5)$$

so that from (3.4) we have

$$\widehat{\beta}_{pol}(s) = \Big[\underbrace{s^{\widehat{q_r}+q_1-1}I_r, \dots, s^{\widehat{q_r}+1}I_r, s^{\widehat{q_r}}I_r}_{rq_1} \Big| \underbrace{s^{\widehat{q_r}-1}I_r \dots, sI_r, I_r}_{r\widehat{q_r}}\Big]$$

$$\times \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ - & - & - & - \\ H_{\widehat{q_r}} & 0 & \dots & 0 \\ H_{\widehat{q_r}-1} & H_{\widehat{q_r}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\widehat{q_r}-(q_1-1)} & H_{\widehat{q_r}-(q_1-2)} & \dots & H_{\widehat{q_r}} \\ \vdots & \vdots & & \vdots \\ H_2 & H_3 & \dots & H_{q_1+1} \\ H_1 & H_2 & \dots & H_{q_1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix}$$

$$\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$= -\left[s^{\widehat{q_r}-1}I_r \dots, sI_r, I_r\right] \begin{bmatrix} H_{\widehat{q_r}} & 0 & \dots & 0 \\ H_{\widehat{q_r}-1} & H_{\widehat{q_r}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\widehat{q_r}-(q_1-1)} & H_{\widehat{q_r}-(q_1-2)} & \dots & H_{\widehat{q_r}} \\ \vdots & \vdots & \ddots & \vdots \\ H_2 & H_3 & \dots & H_{q_1+1} \\ H_1 & H_2 & \dots & H_{q_1} \end{bmatrix}$$

$$\times\begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix}\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \tag{3.6}$$

so that

$$\widehat{\beta}_{pol}(s)=\beta_{\widehat{q}_r-1}s^{\widehat{q}_r-1}+\beta_{\widehat{q}_r-2}s^{\widehat{q}_r-2}+\ldots+\beta_1 s+\beta_0 \tag{3.7}$$

where $\beta_i \in \mathbf{R}^{r\times 1}$, $i=0,1,\ldots,\widehat{q}_r-1$, $\widehat{q}_r \geq 1$ are obtained from the last expression in (3.6).

Taking the inverse Laplace transform of (3.3) we have

$$\beta(t)=L_-^{-1}\left\{\widehat{\beta}(s)\right\}=L_-^{-1}\left\{A(s)^{-1}\widehat{\alpha}(s)\right\}=L_-^{-1}\left\{\widehat{\beta}_{pol}(s)\right\}+L_-^{-1}\left\{\widehat{\beta}_{sp}(s)\right\} \tag{3.8}$$

so that from (3.7) (3.8)

$$L_-^{-1}\left\{\widehat{\beta}_{pol}(s)\right\}=L_-^{-1}\left\{\beta_0+\beta_1 s+\ldots+\beta_{\widehat{q}_r-1}s^{\widehat{q}_r-1}\right\}$$

$$=\beta_0\delta(t)+\beta_1\delta^{(1)}(t)+\ldots+\beta_{\widehat{q}_r-1}\delta^{\left(\widehat{q}_r-1\right)}(t) \tag{3.9}$$

it follows that if $A(s)$ has at least one *zero* at $s=\infty$, i.e. if $\widehat{q}_r\geq 1$ and (as we will see in section 4) *depending on the choice of the initial values* $\beta(0-)$, $\beta^{(1)}(0-),\ldots,\beta^{(q_1-1)}(0-)$, the solution $\beta(t)$ of (1.1) might exhibit an 'impulsive behavior' at $t=0$ which consists of combinations of the Dirac impulse $\delta(t)$ and its $(\widehat{q}_r-1)$-*th* order distributional derivatives as in (3.9) which is associated with the *zeros at* $s=\infty$ of $A(s)$, i.e. due to the fact that in such a case, the natural modes of (1.1), defined as values of s where $A(s)$ loses rank, include also the point at $s=\infty$.

What exactly is meant by the phrase in the previous paragraph '*depending on the choice of the initial values* $\beta(0-)$, $\beta^{(1)}(0-),\ldots,\beta^{(q_1-1)}(0-)$' and how and exactly why such a choice of the initial *values* might give rise to an impulsive behavior in $\beta(t)$ at $t=0$ is fully explained in Remark 2 in Section 4 bellow.

If $A(s)$ has *no zeros at* $s=\infty$ then the Smith-McMillan form of $A(s)$ is polynomial i.e. in this case

$$S_{A(s)}^{\infty}=diag\left[\overbrace{\underbrace{s^{q_1},s^{q_2},\cdots,s^{q_k}}_{k},I_{r-k}}^{r}\right]\in\mathbf{R}[s]^{r\times r},\quad q_1\geq q_2\geq\ldots\geq q_k>0 \tag{3.10}$$

which implies that

$$S_{A(s)^{-1}}^{\infty}=diag\left[I_{r-k},\frac{1}{s^{q_k}},\frac{1}{s^{q_{r-1}}},\ldots,\frac{1}{s^{q_1}}\right]\in\mathbf{R}_{pr}(s)^{r\times r}\Longrightarrow A(s)^{-1}\in\mathbf{R}_{pr}(s)^{r\times r} \tag{3.11}$$

so that the Laurent expansion at $s=\infty$ of $A(s)^{-1}$ will have the form:

$$A(s)^{-1}=H_0+H_{-1}\frac{1}{s}+H_{-2}\frac{1}{s^2}+\ldots \tag{3.12}$$

and thus (2.5) gives

$$\overbrace{\begin{bmatrix} H_0 & 0 & \dots & 0 & 0 \\ H_{-1} & H_0 & \dots & 0 & 0 \\ H_{-2} & H_{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} & H_0 \\ - & - & - & - & - \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} & H_{-1} \\ H_{-(q_1+2)} & H_{-(q_1+1)} & \dots & H_{-3} & H_{-2} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix}}^{(q_1+1)r} \begin{bmatrix} A_{q_1} \\ A_{q_1-1} \\ \vdots \\ A_1 \\ A_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_r \\ - \\ 0 \\ 0 \end{bmatrix} \tag{3.13}$$

Then (3.3) gives

$$\widehat{\beta}(s) = \left[H_0 + H_{-1}\frac{1}{s} + H_{-2}\frac{1}{s^2} + \dots\right]\left[s^{q_1-1}I_r \dots, sI_r, I\right]$$
$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$
$$= \left[\overbrace{s^{q_1-1}I_r, s^{q_1-2}I_r, \dots, sI_r, I_r,}^{q_1 r} \mid \frac{1}{s}I_r, \frac{1}{s^2}I_r \dots\right] \begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_{-1} & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \\ - & - & - & - \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$
$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \tag{3.14}$$

but from (3.13)

$$\begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_{-1} & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} = 0_{q_1 r \times q_1 r}$$

and so from (3.14)

$$\widehat{\beta}(s) = \left[\frac{1}{s}I_r, \frac{1}{s^2}I_r \ldots\right] \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} =: \widehat{\beta}_{sp}(s) \in \mathbf{R}_{pr}(s)^{r\times 1} \quad (3.15)$$

is *strictly proper for every set* of initial values $\beta(0-)$, $\beta^{(1)}(0-)$,...,$\beta^{(q_1-1)}(0-)$. Conversely if $\widehat{\beta}_{pol}(s) = 0$ for every set of initial values $\beta(0-), \beta^{(1)}(0-), \ldots,$ $\beta^{(q_1-1)}(0-)$ then from (3.6) it follows that we must have that

$$\begin{bmatrix} H_{\widehat{q}_r} & 0 & \ldots & 0 \\ H_{\widehat{q}_r-1} & H_{\widehat{q}_r} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\widehat{q}_r-(q_1-1)} & H_{\widehat{q}_r-(q_1-2)} & \ldots & H_{\widehat{q}_r} \\ \vdots & \vdots & \ddots & \vdots \\ H_2 & H_3 & \ldots & H_{q_1+1} \\ H_1 & H_2 & \ldots & H_{q_1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \ldots & A_{q_1-1} \\ 0 & A_0 & \ldots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_0 \end{bmatrix} = 0_{\widehat{q}_r r, q_1 r} \quad (3.16)$$

Now (3.16) implies

$$H_{\widehat{q}_r} \begin{bmatrix} A_0 & A_1 & \ldots & A_{q_1-1} \end{bmatrix} = 0_{r,q_1 r} \quad (3.17)$$

but from (2.5) for $i = \widehat{q}_r + q_1$

$$H_{\widehat{q}_r} A_{q_1} = 0 \quad (3.18)$$

Combining (3.17) and (3.18) gives

$$H_{\widehat{q}_r} \begin{bmatrix} A_0 & A_1 & \ldots & A_{q_1-1} & A_{q_1} \end{bmatrix} = 0_{r,q_1(r+1)}$$

which, since $rank_{\mathbf{R}(s)} A(s) = r \Longrightarrow rank_{\mathbf{R}} \begin{bmatrix} A_0 & A_1 & \ldots & A_{q_1-1} & A_{q_1} \end{bmatrix} = r$, (see Exercise 4.10 in [1]) implies that $H_{\widehat{q}_r} = 0$. Putting $H_{\widehat{q}_r} = 0$ into (3.16) and using similar arguments it can be shown successively that $H_{\widehat{q}_r-1} = \ldots = H_1 = 0$, i.e. that $A(s)^{-1} \in \mathbf{R}_{pr}[s]^{r\times r}$. In view of the above, the absence of impulsive behavior from $\beta(t)$ at $t = 0$ *for every* set of initial values is characterized by the absence from $A(s)$ of *zeros* at $s = \infty$. These facts can be stated as

Theorem 1. *Consider the linear homogeneous matrix differential equation*

$$A(\rho)\beta(t) = 0 \qquad t > 0$$

where $A(\rho) \in \mathbf{R}[\rho]^{r\times r}$, $rank_{\mathbf{R}(\rho)} A(\rho) = r$. *Then* $\beta(t) : (0-, \infty) \to \mathbf{R}^r$ *does not contain impulses at* $t = 0$ *for every set of initial values*

$\beta(0-)$,$\beta^{(1)}(0-)$,...,$\beta^{(q_1-1)}(0-)$, *if and only if the following equivalent conditions hold true:*

- (i) $\widehat{\beta}(s) = A(s)^{-1}\widehat{\alpha}(s) \in \mathbf{R}_{pr}^{r\times 1}(s)$ is *strictly proper.*
- (ii) $\widehat{\beta}_{pol}(s) = 0$
- (iii) $A(s)$ has no zeros at $s = \infty$
- (iv) $A(s)^{-1}$ has no poles at $s = \infty \Longleftrightarrow A(s)^{-1} \in \mathbf{R}_{pr}[s]^{r\times r}$.

Remark 1. *Notice that 'absence of impulsive behavior from $\beta(t)$ at $t = 0$ for every set of initial values does not necessarily imply continuity of $\beta(t)$ at $t = 0$, i.e. we might have $\beta(0-) \neq \beta(0+)$ (see also [5]). A necessary and sufficient condition for the continuity of $\beta(t)$ and its derivatives up to order $q_1 - 1$ at $t = 0$ for every set of initial values $\beta(0-), \beta^{(1)}(0-), \ldots, \beta^{(q_1-1)}(0-)$ is given in Proposition 1 in the following section.*

4. A Closed formula for the solution of the homogenous matrix differential equation $A(\rho)\beta(t) = 0$. Conditions for the continuity of the solution.

From (3.6) and (2.17) and after some algebra (see [1]) we obtain

$$\widehat{\beta}_{pol}(s) = C_{\infty}(sJ_{\infty} - I_{\mu})^{-1} J_{\infty} x_f(0-) \tag{4.1}$$

where

$$x_f(0-) := \begin{bmatrix} B_{\infty} & J_{\infty}B_{\infty} & \ldots & J_{\infty}^{q_1-1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \ldots & A_{q_1-1} \\ 0 & A_0 & \ldots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_0 \end{bmatrix}$$

$$\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathbf{R}^{\mu\times 1} \tag{4.2}$$

also from the second part of (3.4) and (2.18) after some algebra we obtain that

$$\widehat{\beta}_{sp}(s) = C(sI_n - J)^{-1} x_s(0-) \tag{4.3}$$

where

$$x_s(0-) := \begin{bmatrix} J^{q_1-1}B & J^{q_1-2}B & \ldots & B \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix}$$

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathbf{R}^{n\times 1} \tag{4.4}$$

Combining (4.1)(4.3) with (3.4) we finally obtain

$$\widehat{\beta}(s) = \widehat{\beta}_{sp}(s) + \widehat{\beta}_{pol}(s)$$

$$= \begin{bmatrix} C & C_\infty \end{bmatrix} \begin{bmatrix} sI_n - J & 0_{n,\mu} \\ 0_{\mu,n} & sJ_\infty - I_\mu \end{bmatrix}^{-1} \begin{bmatrix} x_s(0-) \\ J_\infty x_f(0-) \end{bmatrix} \tag{4.5}$$

Definition 1. *[1]The vector*

$$\begin{bmatrix} x_s(0-) \\ J_\infty x_f(0-) \end{bmatrix} := \begin{bmatrix} I_n & 0_{n,\mu} \\ 0_{\mu,n} & J_\infty \end{bmatrix} \left[\begin{array}{c|c} J^{q_1-1}B, J^{q_1-2}B, \ldots, B & 0_{n,q_1\mu} \\ \hline 0_{\mu,q_1 n} & B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q_1-1} \end{array} \right]$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \\ A_0 & A_1 & \ldots & A_{q_1-1} \\ 0 & A_0 & \ldots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_0 \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathbf{R}^{(n+\mu)\times 1} \tag{4.6}$$

is defined as the *state at* $t = 0-$ of the homogeneous matrix differential equation $A(\rho)\beta(t) = 0$, $t \geq 0$. $x_s(0-)$ is the *slow state at* $t = 0-$ and $x_f(0-)$ is the *fast state at* $t = 0-$

Taking the inverse Laplace transform of (4.5) we have

$$\beta(t) = L_-^{-1}\left\{\widehat{\beta}(s)\right\}$$

$$= Ce^{Jt}x_s(0-) - C_\infty\left[\delta(t)J_\infty + \delta(t)^{(1)}J_\infty^2 + \ldots + \delta(t)^{\left(\widehat{q}_r-1\right)}J_\infty^{\widehat{q}_r}\right]x_f(0-)$$

So that

$$\beta^{(i)}(t) = CJ^i e^{Jt}x_s(0-)$$

$$-C_\infty\left[\delta(t)^{(i)}J_\infty + \delta(t)^{(i+1)}J_\infty^2 + \ldots + \delta(t)^{\left(i+\widehat{q}_r-1\right)}J_\infty^{\widehat{q}_r}\right]x_f(0-) \quad i = 0,1,2,\ldots \tag{4.7}$$

Since $\delta^{(i)}(t) = 0\ \forall\ t \neq 0$ equations (4.7) for $t = 0+$, give:

$$\beta^{(i)}(0+) = CJ^i x_s(0-) \qquad i = 0,1,2,\ldots \tag{4.8}$$

Writing (4.8) for $i = 0,1,\ldots,q_1-1$ in matrix form we get

$$\begin{bmatrix} \beta(0+) \\ \beta^{(1)}(0+) \\ \vdots \\ \beta^{(q_1-1)}(0+) \end{bmatrix} = \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{q_1-1} \end{bmatrix} x_s(0-) \tag{4.9}$$

Substituting $x_s(0-)$ from (4.4) into (4.9) we obtain that in general the relation between $\beta^{(i)}(0+)$ and $\beta^{(i)}(0-)$ for $i = 0, 1, 2, \ldots q_1 - 1$ is given by

$$
\begin{bmatrix} \beta(0+) \\ \beta^{(1)}(0+) \\ \vdots \\ \beta^{(q_1-1)}(0+) \end{bmatrix}
$$

$$
= \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{q_1-1} \end{bmatrix}
\begin{bmatrix} J^{q_1-1}B & J^{q_1-2}B & \ldots & B \end{bmatrix}
\begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}
$$

$$
= \begin{bmatrix} CJ^{q_1-1}B & CJ^{q_1-2}B & \ldots & CB \\ CJ^{q_1}B & CJ^{q_1-1}B & \ldots & CJB \\ \vdots & \vdots & \ddots & \vdots \\ CJ^{2q_1-2}B & CJ^{2q_1-3}B & \ldots & CJ^{q_1-1}B \end{bmatrix}
\begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}
$$

$$
= \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix}
\begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix}
$$

$$
\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \tag{4.10}
$$

where we made use of eq. (2.18).

Equation (4.10) indicates that if $A(s)$ has zeros at $s = \infty$ and the initial *values* $\beta^{(i)}(0-)$, $i = 0, 1, 2, ..., q_1 - 1$ are chosen arbitrarily then in general $\beta^{(i)}(0+) \neq \beta^{(i)}(0-)$ for $i = 0, 1, 2, \ldots, q_1 - 1$ i.e. there will be a discontinuity in $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \ldots, q_1 - 1$ at $t = 0$. These discontinuities will be described by the given $\beta^{(i)}(0-)$ and the $\beta^{(i)}(0+)$, $i = 0, 1, 2, \ldots, q_1 - 1$ which are obtained from (4.10). If we demand that $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \ldots, q_1 - 1$ are all continuous at $t = 0$ so that $\beta^{(i)}(0-) =$

$\beta^{(i)}(0+)$, $i = 0, 1, 2, \ldots q_1 - 1$ then from (4.10) we see that the given initial values $\beta^{(i)}(0-)$, $i = 0, 1, \ldots, q_1 - 1$ can not be completely arbitrary but must satisfy the relation

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$= \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \tag{4.11}$$

or equivalently for $\beta(t)$ *and* its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \ldots, q_1 - 1$ to be continuous at $t = 0$ the given initial *values* at $t = -0 : \beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$ must satisfy

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$\in \ker \left[I_{rq_1} - \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} \right] \tag{4.12}$$

or equivalently using (2.5)

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \ker \begin{bmatrix} H_0 & H_1 & \cdots & H_{q_1-1} \\ H_{-1} & H_0 & & \vdots \\ \vdots & & \ddots & H_1 \\ H_{-q_1+1} & \cdots & H_{-1} & H_0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_0 & A_1 \\ 0 & \cdots & 0 & A_0 \end{bmatrix} \tag{4.13}$$

Remark 2. *Notice that if $A(s)$ has at least one zero at $s = \infty$ i.e. if $\widehat{q}_r \geq 1$ then this implies that $rank_{\mathbf{R}} A_{q_1} < r$ [1]. In such a case if the initial values at $t = 0-$: $\beta^{(i)}(0-), i = 0, 1, 2, \ldots, q_1 - 1$ are chosen so that (4.13) is not satisfied i.e. if $\beta^{(i)}(0-) \neq \beta^{(i)}(0+), i = 0, 1, 2, \ldots, q_1 - 1$ then steps that result from the components of $\beta(t)$ falling from their initial values in $\beta(0-)$ to the values described by $\beta(0+)$ in (4.10) for $t \geq 0$ are differentiated in accordance with the differential equation $A(\rho)\beta(t) = 0$ giving rise to impulsive behavior in $\beta(t)$ at $t = 0$ according to eq. (3.9). If on the other hand the initial values $\beta^{(i)}(0-)$ are chosen so that (4.13) is satisfied (equivalently condition (4.9) is* *satisfied with* $\beta^{(i)}(0+) = \beta^{(i)}(0-), i = 0,1,2,\ldots,q_1 - 1)$ *i.e. if we demand that* $\beta(t)$ *and its derivatives* $\beta^{(i)}(t)$, $i = 1,2,\ldots,q_1 - 1$ *are continuous at* $t = 0$ *then the 'fast state' at* $t = 0-$: $x_f(0-) = 0$ *(see Proposition 3 bellow) and from (4.1)* $\widehat{\beta}_{pol}(s) = 0$, *so that there will be no impulsive behavior in* $\beta(t)$ *at* $t = 0$. *In this case condition (4.13) imposes certain restrictions on the choice of* $\beta^{(i)}(0-)$, $i = 0,1,2,\ldots,q_1 - 1$.

Remark 3. *Notice that if* $A(s)$ *is monic i.e.* $rank_{\mathbf{R}} A_{q_1} = r$, *then* $A(s)$ *will be both row and column reduced at* $s = \infty$ *[4] and all its row degrees will be equal to* $q_1 \geq 1$. *Consequently* $A(s)$ *will have no zeros at* $s = \infty$ *(i.e.* $q_i = q_1 > 0$, $i = 1,2,\ldots,r$*) and its row degrees* q_1 *will be the orders of its poles at* $s = \infty$, *so that the Smith-McMillan form at* $s = \infty$: $S^{\infty}_{A(s)}$ *of* $A(s)$ *will be given by* $S^{\infty}_{A(s)} = diag\left[s^{q_1}, s^{q_1}, \ldots s^{q_1}\right] = s^{q_1} I_r$. *This in turn implies that* $A(s)^{-1} \in \mathbf{R}_{pr}(s)^{r\times r}$ *will be strictly proper with Smith-McMillan form at* $s = \infty : S^{\infty}_{A(s)^{-1}} = \left[S^{\infty}_{A(s)}\right]^{-1} = \frac{1}{s^{q_1}} I_r$. *So in this case the Laurent expansion of* $A(s)^{-1}$ *at* $s = \infty$ *will 'start' from the term* $\frac{1}{s^{q_1}} H_{-q_1}$ *i.e.* $H_0 = H_{-1} = H_{-2} = \ldots = H_{-(q_1-1)} = 0$ *and* $H_{-q_1} \neq 0$, *so that* $A(s)^{-1} = \frac{1}{s^{q_1}} H_{-q_1} + \frac{1}{s^{q_1+1}} H_{-(q_1+1)} + \ldots$ *Due to this fact if* $rank_{\mathbf{R}} A_{q_1} = r$ *condition (4.11) becomes*

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} = \begin{bmatrix} H_{-q_1} & 0 & \ldots & 0 \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} \times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \tag{4.14}$$

which is an identity since from (2.5) we have that

$$\begin{bmatrix} H_{-q_1} & 0 & \ldots & 0 \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} = I_{q_1 r} \tag{4.15}$$

Conversely if for every $\beta^{(i)}(0-)$, $i=0,1,2,\ldots,q_1-1$, $\beta^{(i)}(0-)=\beta^{(i)}(0+)$ from (4.10) it follows that we must have

$$\begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \ldots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \ldots & H_{-q_1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} = I_{rq_1} \quad (4.16)$$

which implies that both block matrices in (4.16) must have full rank. From the second block lower triangular matrix in (4.16) this in turn implies that $rank_{\mathbf{R}} A_{q_1}=r$ i.e. that $A(s)$ is monic. These considerations give rise to

Proposition 1. *There is no discontinuity in $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i=1,2,\ldots q_1-1$ at $t=0$ i.e. $\beta^{(i)}(0-)=\beta^{(i)}(0+)$, $i=0,1,2,\ldots,q_1-1$ for every set of initial values $\beta^{(i)}(0-)$, $i=1,2,\ldots,q_1-1$ iff $A(s)$ is monic i.e. iff $rank_{\mathbf{R}} A_{q_1}=r$.*

If $A(s)^{-1}\in\mathbf{R}_{pr}(s)^{r\times r}$ and we consider only the first equation in (4.11) we obtain

$$\beta(0+)$$

$$= Cx_s(0-) = C\begin{bmatrix} J^{q_1-1}B & J^{q_1-2}B & \ldots & B \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$= \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \quad (4.17)$$

But from (2.5) we have

$$\begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \ldots & H_{-1} & | & H_0 \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \ldots & 0 \\ A_{q_1-1} & A_{q_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \ldots & A_{q_1} \\ - & - & - & - \\ A_0 & A_1 & \ldots & A_{q_1-1} \end{bmatrix} \quad (4.18)$$

$$= \begin{bmatrix} I_r & 0 & 0 & \ldots & 0 \end{bmatrix}$$

which can be written as

$$\left[\begin{array}{cccc} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \end{array}\right]\left[\begin{array}{cccc} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{array}\right] \tag{4.19}$$

$$= \left[\begin{array}{ccccc} I_r & 0 & 0 & \dots & 0 \end{array}\right] - H_0\left[\begin{array}{cccc} A_0 & A_1 & \dots & A_{q_1-1} \end{array}\right]$$

so that (4.17) gives

$$\beta(0+) = \beta(0-) - H_0\left[A_0\beta(0-) + A_1\beta^{(1)}(0-) + \ldots + A_{q_1-1}\beta^{(q_1-1)}(0-)\right] \tag{4.20}$$

which implies that if $H_0 = 0$ i.e. if $A(s)^{-1}$ is *strictly proper* then $\beta(t)$ (but not necessarily its derivatives) is continuous at $t = 0$, i.e. $\beta(0+) = \beta(0-)$. Conversely if we require that $\beta(0+) = \beta(0-)$ for every set of initial *values* $\beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$then from (4.20) it follows that we must have

$$H_0\left[\begin{array}{cccc} A_0 & A_1 & \dots & A_{q_1-1} \end{array}\right] = 0 \tag{4.21}$$

but from (2.5)

$$H_0 A_{q_1} = 0 \tag{4.22}$$

Now (4.21) and (4.22) can be written as

$$H_0\left[\begin{array}{ccccc} A_0 & A_1 & \dots & A_{q_1-1} & A_{q_1} \end{array}\right] = 0 \tag{4.23}$$

which again, since $rank_{\mathbf{R}(s)}A(s) = r \Longrightarrow rank_{\mathbf{R}}\left[\begin{array}{ccccc} A_0 & A_1 & \dots & A_{q_1-1} & A_{q_1} \end{array}\right] = r$, implies that $H_0 = 0$, i.e. $A(s)^{-1}$ is *strictly proper*. The above argument gives rise to the following

Proposition 2. *If $A(s)^{-1} \in \mathbf{R}_{pr}(s)^{r\times r}$ then $\beta(t)$ (but not necessarily its derivatives) is continuous at $t = 0$, i.e. $\beta(0+) = \beta(0-)$ for every set of initial values at $t = 0- : \beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$ iff $H_0 = 0$ i.e. iff $A(s)^{-1}$ is strictly proper (compare this with Theorem 1).*

Remark 4. *Similarly it can be shown that if $A(s)^{-1} \in \mathbf{R}_{pr}(s)^{r\times r}$*

$$\beta^{(1)}(0+) = \beta^{(1)}(0-) - H_{-1}\left[A_0\beta(0-) + A_1\beta^{(1)}(0-) + \ldots + A_{q_1-1}\beta^{(q_1-1)}(0-)\right]$$

so that $\beta^{(1)}(t)$ is continuous at $t = 0$, i.e. $\beta^{(1)}(0+) = \beta^{(1)}(0-)$ for every set of initial values at $t = 0- : \beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$ iff $H_{-1} = 0$. This result can be generalized by showing that continuity at $t = 0$ of all derivatives $\beta^{(i)}(t)$ of $\beta(t)$ up to order $j \leq q_1 - 1$ and for every set of initial values at $t = 0- : \beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$ is guaranteed iff $H_0 = H_{-1} = H_{-2} = \ldots = H_{-j} = 0$. (Note that $j \leq q_1 - 1$ because otherwise the conditions $H_0 = H_{-1} = H_{-2} = \ldots = H_{-(q_1-1)} = H_{-q_1} = 0$ would imply that $A(s)$ has a pole at $s = \infty$ of order greater than q_1).

Finally we state

Proposition 3. *Assume that the given initial values* $\beta^{(i)}(0-)$, $i = 0, 1, 2, \ldots, q_1 - 1$ *satisfy (4.13) so that* $\beta(t)$ *and its derivatives* $\beta^{(i)}(t), i = 1, 2, \ldots, q_1 - 1$ *are all continuous at* $t = 0$ *i.e.* $\beta^{(i)}(0-) = \beta^{(i)}(0+) =: \beta^{(i)}(0)$, $i = 0, 1, 2, \ldots, q_1 - 1$. *Then* $x_f(0-) = 0$ *and the solution of (1.1) is given by* $\beta(t) = Ce^{Jt}x_s(0-)$ *where* $x_s(0-)$ *is given by (4.4)*

Proof. [1].

Example 1. *Consider the system of differential equations*

$$\begin{array}{c} \dot{\beta}_1(t) + \dddot{\beta}_2(t) = -\beta_1(t) \\ \dot{\beta}_2(t) = -\beta_2(t) \end{array} \qquad t \geq 0$$

which can be written in matrix form as

$$\begin{bmatrix} \rho+1 & \rho^3 \\ 0 & \rho+1 \end{bmatrix} \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $A(\rho)\beta(t) = 0, \beta(t) := \begin{bmatrix} \beta_1(t) & \beta_2(t) \end{bmatrix}^\top, r = 2, q = 3$ *where*

$$A(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\rho + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\rho^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\rho^3$$

Now the Smith McMillan form of $A(s)$ *at* $s = \infty$ *is* $S^{\infty}_{A(s)} = diag\left[s^3, 1/s\right]$, *i.e.* $A(s)$ *has a pole at* $s = \infty$ *of order* $q = q_1 = 3$ *and a zero at* $s = \infty$ *of order* $\widehat{q}_2 = 1$ *and thus* $A(s)^{-1}$ *is a non-proper rational matrix*:

$$\begin{aligned} A(s)^{-1} &= \begin{bmatrix} \frac{1}{s+1} & \frac{-s^3}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+1} & -\frac{(3s+2)}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} + \begin{bmatrix} 0 & 2-s \\ 0 & 0 \end{bmatrix} = H_{sp}(s) + H_{pol}(s) \end{aligned}$$

from which we obtain that $H_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, $H_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $H_j = 0_{2,2}$ *for* $j > 1$, *and by long division* $\frac{1}{s+1} = 1s^{-1} - 1s^{-2} + 1s^{-3} + \ldots$, $\frac{-(3s+2)}{(s+1)} = -3s^{-1} + 4s^{-2} - 5s^{-3} + \ldots$ *i.e.*

$$H_{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \; H_{-2} = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix}, \; H_{-3} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \ldots$$

From condition 4.13 for $\beta(t)$ *and its derivatives* $\beta^{(i)}(t)$, $i = 1, 2$ *to be con-*

tinuous at $t = 0$ *so that* $\beta^{(i)}(0-) = \beta^{(i)}(0+)$, $i = 0, 1, 2$ *the initial values at* $t = 0-$,$\beta_j^{(i)}(0-)$, $j = 1, 2,\ i = 0, 1, 2$ *must satisfy*

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} \in \ker \begin{bmatrix} H_0 & H_1 & H_2 \\ H_{-1} & H_0 & H_1 \\ H_{-2} & H_{-1} & H_0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & A_2 \\ 0 & A_0 & A_1 \\ 0 & 0 & A_0 \end{bmatrix}$$

$$= \ker \begin{bmatrix} 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 1 & -3 & 0 & 2 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \ker \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 4 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis for the right kernel of the above matrix is: $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ *and*

thus we must have that

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} = \begin{bmatrix} \beta_1(0-) \\ \beta_2(0-) \\ \beta_1^{(1)}(0-) \\ \beta_2^{(1)}(0-) \\ \beta_1^{(2)}(0-) \\ \beta_2^{(2)}(0-) \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \alpha, \beta \in \mathbf{R} \qquad (4.24)$$

from which we obtain that $\alpha = \beta_2(0-)$, $\beta = \beta_1(0-) - 2\beta_2(0-)$ *so that from 4.24 we obtain that* $\beta_j^{(i)}(0-)$, $j = 1, 2,\ i = 0, 1, 2$ *must satisfy the conditions*

$$\beta_1^{(1)}(0-) = -\beta_1(0-) + \beta_2(0-) \qquad (4.25)$$

$$\beta_2^{(1)}(0-) = -\beta_2(0-) \qquad (4.26)$$

$$\beta_1^{(2)}(0-) = \beta_1(0-) - 2\beta_2(0-) \qquad (4.27)$$

$$\beta_2^{(2)}(0-) = \beta_2(0-) \qquad (4.28)$$

An *irreducible at* $s=\infty$ generalized state space realization of the polynomial part $H_{pol}(s)=\begin{bmatrix}0 & 2-s\\ 0 & 0\end{bmatrix}$ of $A(s)^{-1}$ is given by the triple

$$C_\infty=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ J_\infty=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ B_\infty=\begin{bmatrix}0 & -1\\ 0 & 2\end{bmatrix}$$

Formula (4.2) gives

$$x_f(0-)=\begin{bmatrix}B_\infty, & J_\infty B_\infty, & J_\infty^2 B_\infty\end{bmatrix}\begin{bmatrix}A_0 & A_1 & A_2\\ 0 & A_0 & A_1\\ 0 & 0 & A_0\end{bmatrix}\begin{bmatrix}\beta(0-)\\ \beta^{(1)}(0-)\\ \beta^{(2)}(0-)\end{bmatrix}$$

$$=\begin{bmatrix}0 & -1 & | & 0 & 0 & | & 0 & 0\\ 0 & 0 & | & 0 & -1 & | & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & | & 1 & 0 & | & 0 & 0\\ 0 & 1 & | & 0 & 1 & | & 0 & 0\\ - & - & | & - & - & | & - & -\\ 0 & 0 & | & 1 & 0 & | & 1 & 0\\ 0 & 0 & | & 0 & 1 & | & 0 & 1\\ - & - & | & - & - & | & - & -\\ 0 & 0 & | & 0 & 0 & | & 1 & 0\\ 0 & 0 & | & 0 & 0 & | & 0 & 1\end{bmatrix}$$

$$\times\begin{bmatrix}\beta_1(0-)\\ \beta_2(0-)\\ -\\ \beta_1^{(1)}(0-)\\ \beta_2^{(1)}(0-)\\ -\\ \beta_1^{(2)}(0-)\\ \beta_2^{(2)}(0-)\end{bmatrix}$$

$$=\begin{bmatrix}-\beta_2(0-)-\beta_2^{(1)}(0-)\\ 2\beta_2(0-)+\beta_2^{(1)}(0-)-\beta_2^{(2)}(0-)\end{bmatrix}\overset{4.25-4.28}{=}\begin{bmatrix}0\\ 0\end{bmatrix}\tag{4.29}$$

i.e. we have that $x_f(0-)=0$ and thus from (4.1)

$$\widehat{\beta}_{pol}(s)=C_\infty(sJ_\infty-I_\mu)^{-1}J_\infty x_f(0-)=0$$

as in Proposition 3 so that $\beta_\infty(t):=L_-^{-1}\left\{\widehat{\beta}_{pol}(s)\right\}=0$, and there is no impulsive behavior in $\beta(t)$ at $t=0$.
A minimal realization C,J,B of the strictly proper part of $A(s)^{-1}$:

$$H_{sp}(s)=\begin{bmatrix}\frac{1}{s+1} & -\frac{3s+2}{(s+1)^2}\\ 0 & \frac{1}{s+1}\end{bmatrix}$$

is given by $C = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $(n = 2)$ and formula (4.4) gives

$$x_s(0-) := \begin{bmatrix} J^2B & JB & B \end{bmatrix} \begin{bmatrix} A_3 & 0 & 0 \\ A_2 & A_3 & 0 \\ A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & | & -1 & 1 & | & 1 & 0 \\ 0 & 1 & | & 0 & -1 & | & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 1 & | & 0 & 0 & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 & | & 0 & 0 \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 1 & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 & | & 0 & 0 \\ - & - & | & - & - & | & - & - \\ 1 & 0 & | & 0 & 0 & | & 0 & 1 \\ 0 & 1 & | & 0 & 0 & | & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1(0-) \\ \beta_2(0-) \\ - \\ \beta_1^{(1)}(0-) \\ \beta_2^{(1)}(0-) \\ - \\ \beta_2^{(1)}(0-) \\ \beta_2^{(2)}(0-) \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1(0-) + \beta_2(0-) - \beta_2^{(1)}(0-) + \beta_2^{(2)}(0-) \\ \beta_2(0-) \end{bmatrix} = \begin{bmatrix} x_{s1}(0-) \\ x_{s2}(0-) \end{bmatrix}$$

which due to the constraints 4.25-4.12 gives that

$$x_s(0-) = \begin{bmatrix} x_{s1}(0-) \\ x_{s2}(0-) \end{bmatrix} = \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix} = x(0)$$

and thus the solution of the d.e. is

$$\begin{aligned} \beta(t) &= \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} = \mathcal{L}^{-1}\left\{\widehat{\beta}_{sp}(s)\right\} = \mathcal{L}^{-1}\left\{C(sI_n - J)^{-1} x_s(0-)\right\} \\ &= Ce^{Jt}x_s(0-) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & te^{-t} - 3e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix} \end{aligned}$$

i.e.

$$\begin{matrix} \beta_1(t) = \beta_1(0-)e^{-t} + \beta_2(0-)te^{-t} \\ \beta_2(t) = \beta_2(0-)e^{-t} \end{matrix} \qquad t \geq 0$$

which for $t = 0+$ and due to conditions 4.25-4.12 gives that $\beta_1^{(j)}(0+) = \beta_1^{(j)}(0-)$, $\beta_2^{(j)}(0+) = \beta_2^{(j)}(0-)$, $j = 0, 1, 2$ i.e that $\beta(t)$,$\beta^{(1)}(t)$, $\beta^{(2)}(t)$ are continuous at $t = 0$.

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