Fundamental Equivalence of Discrete-Time AR Representations

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Abstract

We examine the problem of equivalence of discrete time auto-regressive representations (DTARRs) over a finite time interval. Two DTARRs are defined as *fundamentally equivalent* (FE) over a finite time interval $[0, N]$ if their solution spaces or behaviours are isomorphic. We generalise the concept of strict equivalence (SE) of matrix pencils to the case of general polynomial matrices and in turn we show that FE of DTARRs implies SE of the underlying polynomial matrices.

1. Introduction

The problem of equivalence of continuous-time linear systems has been the subject of several studies in recent years and many definitions have been proposed. There are two main approaches to the problem of system equivalence. The first one requires preservation of the structural invariants of the matrices describing the systems, while the second deals with the relations of the solution sets or behaviors of the corresponding systems. Both approaches seem to be equally powerful and in certain cases ([22], [18], [17]) they are proved to be equivalent.

An equivalence relation based on the preservation of structural invariants first appears in [23], where *strict system equivalence* was introduced. Strict system equivalence requires systems to have the same finite frequency structure. This requirement is justified by the fact that systems with the same finite structure will exhibit the same smooth behavior, while in [22] it is shown that strict system equivalence implies the existence of an isomorphism between the smooth solution spaces of the systems involved. In an attempt to generalize strict system equivalence to preserve both the finite and infinite system structures Verghese [27] proposed, in the context of generalized state space systems, the notion of *strong equivalence* which took on a closed form description in [21] as *complete system equivalence*.

On the other hand, the behavioral approach (see [24], [25], [26]) starts from the requirement of existence of an isomorphism between the behaviors of the systems under investigation. However, this approach at least in its original form is only concerned with the smooth behavior of the underlying systems.

In the discrete-time case the question of equivalence is not that clear. The first thing to notice is that in case we are interested only in proper (causal) discrete-time systems, the problem of equivalence and the corresponding theory can be easily transferred from the continuous time context to the discrete one and both approaches can be employed equally well. However, when non-proper or singular discrete-time systems come in to focus, the situation becomes more complicated. Singular discrete-time systems, exhibit non-causal behavior and the natural framework for their study appears to be a finite time interval where the system can be essentially decomposed into a purely causal and a purely anti-causal part (see [1], [2], [3], [4], [5], [7]). The causal and anti-causal behavior of such systems have been proved to be associated to the finite and infinite elementary divisors structure of the matrices describing the systems (see [3] for descriptor systems or [6], [16] for higher order systems).

In this paper we address the question of equivalence of discrete-time systems described by auto-regressive representations (DTARRs) in the framework discussed above and propose an approach based initially on the preservation of the structural invariants of the polynomial matrices involved, which in turn provides the background to introduce a behaviour-oriented system equivalence.

We consider systems of the form

$$A_q\xi_{k+q} + A_{q-1}\xi_{k+q-1} + ... + A_0\xi_k = 0 \tag{1.1}$$

where $A_i \in \mathbb{R}^{r\times r}, i = 0, 1, ..., q$, over a *finite* time interval $k = 0, 1, ..., N$. Two DTARRs are defined as *fundamentally equivalent* (FE) if their solution spaces or behaviors are isomorphic in a particular way. Motivated by the fact that the behavior of the DTARR (1.1), when considered over a *finite* time interval $[0, N]$, depends on the algebraic structure of both the *finite* and the *infinite elementary divisors* of the polynomial matrix $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \mathbb{R}[\sigma]^{r\times r}$ associated with (1.1) [6], [16], we show that this structure is identical with the corresponding structure of a block companion matrix pencil $\sigma\overline{E} - \overline{A} \in \mathbb{R}[\sigma]^{rq\times rq}$ which constitutes a *linearization* of the polynomial matrix [9] and consequently the DTARR associated with $\sigma\overline{E} - \overline{A}$ constitutes the natural first-order *fundamentally equivalent* representation (realization) of (1.1). As a result we propose a generalization of the concept of *strict equivalence* (SE) of regular matrix pencils [8] to the case of general nonsingular polynomial matrices. Finally, as a consequence of our results we show that two DTARRs described by nonsingular polynomial matrices of possibly different degrees and dimensions are FE if and only if these polynomial matrices are SE.

The paper is organized as follows. Section 2 we provides the necessary mathematical background for the consequent sections. In Section 3 a generalization of strict equivalence to the polynomial matrix case is introduced and certain algebraic results are obtained. In Section 4 we propose the notion of FE of DTARRs and provide the main results of the paper. Finally, in Section 5 we summarize our results and propose directions for further research on the subject.

2. Mathematical Background

In what follows $\mathbb{R}, \mathbb{C}$ denote respectively the fields of real and complex numbers and $\mathbb{Z}^+$ denotes the non negative integers. By $\mathbb{R}(\sigma)^{p\times m}$, $\mathbb{R}_{pr}(\sigma)^{p\times m}$ and $\mathbb{R}[\sigma]^{p\times m}$ we denote the sets of $p \times m$ rational, proper rational and polynomial matrices respectively with real coefficients and indeterminate σ. Let

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \mathbb{R}[\sigma]^{r\times r} \tag{2.1}$$

with $rank_{\mathbb{R}(\sigma)}A(\sigma) = r$. The (finite) zeros of $A(\sigma)$ are the roots of the equation $\det A(\sigma) = 0$, equivalently $\lambda_i \in \mathbb{C}$ is a (finite) *zero* of $A(\sigma)$ iff $rank_{\mathbb{C}}A(\lambda_i) < r$. Assume that $A(\sigma)$ has l distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$, and let

$$S^{\lambda_i}_{A(\sigma)} = diag\{(\sigma - \lambda_i)^{m_{i1}}, ..., (\sigma - \lambda_i)^{m_{ir}}\} \tag{2.2}$$

be the local Smith form of $A(\sigma)$ at $\sigma = \lambda_i, i = 1, 2, ..., l$ where $m_{ij} \in \mathbb{Z}^+$ and $0 \leq m_{i1} \leq m_{i2} \leq ... \leq m_{ir}$. The terms $(\sigma - \lambda_i)^{m_{ij}}$ are called the *finite elementary divisors (FEDs)* of $A(\sigma)$ at $\sigma = \lambda_i$. The non-negative integers m_{ij}, $j = 1, 2, ..., r$ are the partial multiplicities of λ_i and $m_i := \sum_{j=1}^{r} m_{ij}, i = 1, 2, ..., l$ is the multiplicity of λ_i. Let $m_{ij} > 0$ and

$$J_{ij} := \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_{ij}\times m_{ij}}, \quad \begin{matrix} i = 1, 2, \ldots, l \\ j = 1, 2, ..., r \end{matrix} \tag{2.3}$$

be the Jordan block corresponding to the finite elementary divisors $(\sigma - \lambda_i)^{m_{ij}}$ of $A(\sigma)$ and

$$J_i := block\ diag \begin{bmatrix} J_{i1}, & J_{i,2}, & \ldots & , J_{ir} \end{bmatrix} \in \mathbb{R}^{m_i\times m_i}, \quad i = 1, 2, \ldots, l \tag{2.4}$$

A pair of matrices $C_i \in \mathbb{R}^{r\times m_i}$, $J_i \in \mathbb{R}^{m_i\times m_i}$ is called a (finite) *Jordan pair* of $A(\sigma)$ corresponding to the zero of $A(\sigma)$ at $\sigma = \lambda_i$ [9](Theorem 7.1 page 184) iff

$$A_qC_iJ_i^q + A_{q-1}C_iJ_i^{q-1} + \ldots + A_1C_iJ_i + A_0C_i = 0_{r,m_i} \tag{2.5}$$

and

$$rank_{\mathbb{R}} \begin{bmatrix} C_i \\ C_iJ_i \\ \vdots \\ C_iJ_i^{m_i-1} \end{bmatrix} = m_i \tag{2.6}$$

Let $n := \deg(\det A(\sigma)) = \sum_{i=1}^{l} m_i$. The pair of matrices

$$C_F := [C_1, C_2, ..., C_l] \in \mathbb{R}^{r\times n}, \qquad J_F := diag\{J_1, J_2, ..., J_l\} \in \mathbb{R}^{n\times n} \tag{2.7}$$

is defined as a *finite Jordan pair* [9] of $A(\sigma)$ and satisfies the following conditions

$$A_qC_FJ_F^q + A_{q-1}C_FJ_F^{q-1} + \ldots + A_1C_FJ_F + A_0C_F = 0_{r,n} \tag{2.8}$$

$$rank_{\mathbb{R}} \begin{bmatrix} C_F \\ C_FJ_F \\ \vdots \\ C_FJ_F^{n-1} \end{bmatrix} = n \tag{2.9}$$

The *dual* matrix $\tilde{A}(\sigma)$ of $A(\sigma)$ [9], [12] is defined as $\tilde{A}(\sigma) := \sigma^q A(\sigma^{-1}) = A_0\sigma^q + A_1\sigma^{q-1}+\ldots+A_q$. Since $rank\tilde{A}(0) = rankA_q$, $\tilde{A}(\sigma)$ has zeros at $\sigma = 0$ iff $rankA_q < r$. Let

$$S^0_{\tilde{A}(\sigma)} = diag\{\sigma^{\mu_1}, ..., \sigma^{\mu_r}\} \tag{2.10}$$

be the local Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$ where $\mu_j \in \mathbb{Z}^+$ and $0 \le \mu_1 \le \mu_2 \le ... \le \mu_r$. The *infinite elementary divisors (IEDs)* of $A(\sigma)$ are defined as the finite elementary divisors σ^{μ_j} of its dual $\tilde{A}(\sigma)$ at $\sigma = 0$. Let $J_{j\infty} \in \mathbb{R}^{\mu_j\times\mu_j}, j = 1, 2, ..., r$ be the Jordan block corresponding to the non-trivial finite elementary divisors σ^{μ_j}, of $\tilde{A}(\sigma)$, $\mu := \sum_{j=1}^{r}\mu_j$ and

$$J_\infty := block\ diag\left[\ J_{1\infty}, \quad J_{2\infty}, \quad \ldots \quad , J_{r\infty}\ \right] \in \mathbb{R}^{\mu\times\mu} \tag{2.11}$$

A *finite Jordan pair* $C_\infty \in \mathbb{R}^{r\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu}$ of the dual matrix $\tilde{A}(\sigma)$ corresponding to the zero of $\tilde{A}(\sigma)$ at $\sigma = 0$ is defined as an *infinite Jordan pair* of $A(\sigma)$ and according to (2.8) and (2.9) satisfies the following conditions

$$A_0C_\infty J^q_\infty + A_1C_\infty J^{q-1}_\infty + \ldots + A_qC_\infty = 0_{r,\mu} \tag{2.12}$$

$$rank_{\mathbb{R}}\begin{bmatrix} C_\infty \\ C_\infty J_\infty \\ \vdots \\ C_\infty J^{\mu-1}_\infty \end{bmatrix} = \mu \tag{2.13}$$

Let

$$S^\infty_{A(\sigma)}(\sigma) = diag\left[\underbrace{\sigma^{q_1}, \ldots, \sigma^{q_k}, I_{v-k}}_{v}, \underbrace{\frac{1}{\sigma^{\widehat{q}_{v+1}}}, \ldots, \frac{1}{\sigma^{\widehat{q}_r}}}_{r-v}\right] \tag{2.14}$$

be the Smith-McMillan form of $A(\sigma)$ at $\sigma = \infty$ [12] where

$$q_1 \ge q_2 \ge \ldots \ge q_k > 0 = q_{k+1} = \ldots = q_v \tag{2.15}$$

$$\widehat{q}_r \ge \widehat{q}_{r-1} \ge \ldots \ge \widehat{q}_{v+1} > 0 \tag{2.16}$$

are respectively the orders of the *poles* and the *zeros* at $\sigma = \infty$ of $A(\sigma)$. Then it is proved in [19] that $q_1 = q$ and in [12] that the local Smith form $S^0_{\tilde{A}(\sigma)}(\sigma)$ of $\tilde{A}(\sigma)$ at $\sigma = 0$ is given by

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \sigma^{q_1}S^\infty_{A(\sigma)}(\sigma^{-1}) = diag\left[\underbrace{\overbrace{1, \sigma^{q_1-q_2}, \ldots, \sigma^{q_1-q_k}}^{k}, \sigma^{q_1}I_{v-k}}_{v}, \sigma^{q_1+\widehat{q}_{v+1}}, \ldots, \sigma^{q_1+\widehat{q}_r}\right] \tag{2.17}$$

so that the orders μ_j of the IEDSs $\sigma^{\mu_j}, j = 1, 2, ..., r$ of $A(\sigma)$ are given by

$$\mu_1 = 0 \tag{2.18}$$

$$\mu_j = q_1 - q_j \qquad j = 2, 3, \ldots, k \tag{2.19}$$

$$\mu_j = q_1 \qquad j = k+1, \ldots, v \tag{2.20}$$

$$\mu_j = q_1 + \widehat{q}_j \qquad j = v+1, \ldots, r \tag{2.21}$$

and thus we have

Proposition 2.1. *The total number of elementary divisors (finite and infinite ones and multiplicities accounted for) of $A(\sigma)$ is equal to the product rq, where r is the dimension and q is the degree of $A(\sigma)$, i.e.*

$$n + \mu = rq \tag{2.22}$$

Proof. Assume that the Smith-McMillan form of $A(\sigma)$ at $\sigma = \infty$ is given by (2.14). To prove (2.22) use (2.18), (2.19), (2.20), (2.21) in conjunction with the fact that for any nonsingular polynomial matrix $A(\sigma)$ the total number of zeros of $A(\sigma)$ in $\mathbb{C} \cup \{\infty\}$ = total number of poles of $A(\sigma)$ at infinity (see [12]). ■

3. Strict Equivalence of polynomial matrices

In this section we investigate the finite and infinite elementary divisors structure of regular polynomial matrices. The elementary divisors structure of a polynomial matrix plays a very important role in the study of the behaviour of DTARR's (see [3], [6]). Our goal for this section is to introduce an equivalence relation for polynomial matrices that preserves both FED and IED structures. This can be done by extending the notion of strict equivalence, originally introduced for matrix pencils, to the polynomial case.

Before we proceed to the analysis that will eventually lead us to the generalization of strict equivalence, it is important to notice the following

Lemma 3.1. *Let $A_i(\sigma) = A_{i,q^i}\sigma^{q^i} + ... + A_{i1}\sigma + A_{i0} \in \mathbb{R}[\sigma]^{r_i \times r_i}$, $i = 1, 2$ be two non-singular polynomial matrices. If $A_1(\sigma)$, $A_2(\sigma)$ have identical (non-trivial) FED and IED structures then $r_1 q^1 = r_2 q^2$.*

Proof. This is direct consequence of (2.22). ■

The above result justifies the following definition.

Definition 3.2. *Let $p \in \mathbb{Z}^+$. Then $\mathbb{R}_p[\sigma]$ will denote the set of polynomial matrices $A(\sigma) = A_q\sigma^q + ... + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{r \times r}$, satisfying $rq = p$.*

Consider now a polynomial matrix as in (2.1) and the associated regular matrix pencil

$$\sigma\overline{E} - \overline{A} := \begin{bmatrix} \sigma I_r & -I_r & \dots & 0 & 0 \\ 0 & \sigma I_r & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma I_r & -I_r \\ A_0 & A_1 & \dots & A_{q-2} & \sigma A_q + A_{q-1} \end{bmatrix} \in \mathbb{R}[\sigma]^{rq \times rq} \tag{3.1}$$

The above pencil is known as a *block companion linearization* of a nonsingular polynomial matrix $A(\sigma)$ [9]. It is well known (see [9]) that the polynomial matrix $A(\sigma)$ and its associated block companion form posses identical finite elementary divisor structures. We generalize this result to relate the infinite elementary divisor structures of $A(\sigma)$ and $\sigma\overline{E} - \overline{A}$.

Lemma 3.3. *Let $A(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$ as in (2.1). Then the matrix pencil $\sigma\overline{E} - \overline{A} \in \mathbb{R}[\sigma]^{rq \times rq}$ in (3.1) and $A(\sigma)$, have identical nontrivial FEDs and IEDs.*

Proof. The fact that $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ and $\sigma\overline{E}-\overline{A} \in \mathbb{R}\left[\sigma\right]^{rq\times rq}$ have identical FEDs is well known [9] (section 7.2 page 186). We shall prove that $\sigma\overline{E}-\overline{A}$ and $A(\sigma)$, have the same non-trivial IEDs. Consider the matrix pencil in (3.1) and the biproper rational matrices:

$$U_{\infty}(\sigma) := \begin{bmatrix} I_r & 0 & \cdots & 0 & 0 \\ 0 & I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_r & 0 \\ -W_0(\sigma) & -W_1(\sigma) & \cdots & -W_{q-2}(\sigma) & I_r \end{bmatrix} \in \mathbb{R}_{pr}(\sigma)^{qr\times qr} \tag{3.2}$$

$$V_{\infty}(\sigma) := \begin{bmatrix} I_r & I_r\sigma^{-1} & I_r\sigma^{-2} & \cdots & I_r\sigma^{-(q-2)} & I_r\sigma^{-(q-1)} \\ 0 & I_r & I_r\sigma^{-1} & \cdots & I_r\sigma^{-(q-3)} & I_r\sigma^{-(q-2)} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & I_r & I_r\sigma^{-1} \\ 0 & 0 & 0 & \cdots & 0 & I_r \end{bmatrix} \in \mathbb{R}_{pr}(\sigma)^{qr\times qr} \tag{3.3}$$

where $W_i(\sigma) = \sigma^{-(i+1)}\sum_{k=0}^{i} A_k\sigma^k \in \mathbb{R}_{pr}(\sigma)^{r\times r}$, $i = 0, 1, 2, \ldots q-2$. By straightforward computation it is easy to verify that

$$U_{\infty}(\sigma)(\sigma\overline{E}-\overline{A})V_{\infty}(\sigma) = diag\{\sigma I_{r(q_1-1)}, \frac{1}{\sigma^{q_1-1}}A(\sigma)\} \tag{3.4}$$

and since $U_{\infty}(\sigma)$ and $V_{\infty}(\sigma)$ are biproper rational matrices, the pencil $\sigma\overline{E}-\overline{A}$ and the block diagonal matrix in the right-hand side of (3.4), are equivalent at $\sigma = \infty$ [12], hence they have identical pole-zero structures at $\sigma = \infty$.
Assume now that $T_L(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$, $T_R(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$ are biproper rational matrices that bring the matrix $A(\sigma)$ to its Smith-McMillan form $S^{\infty}_{A(\sigma)}(\sigma)$ at $\sigma = \infty$, i.e. let that

$$T_L(\sigma)A(\sigma)T_R(\sigma) = S^{\infty}_{A(\sigma)}(\sigma) = diag\left[\underbrace{\sigma^{q_1}, \ldots, \sigma^{q_k}, I_{v-k}}_{v}, \underbrace{\frac{1}{\sigma^{\hat{q}_{v+1}}}, \ldots, \frac{1}{\sigma^{\hat{q}_r}}}_{r-v}\right] \tag{3.5}$$

By pre and post-multiplying (3.4), respectively by $\hat{T}_L(\sigma) := diag\left[I_{r(q_1-1)}, T_L(\sigma)\right] \in \mathbb{R}_{pr}(\sigma)^{qr\times qr}$ and $\hat{T}_R(\sigma) := diag\left[I_{r(q_1-1)}, T_R(\sigma)\right] \in \mathbb{R}_{pr}(\sigma)^{qr\times qr}$ we have:

$$\hat{T}_L(\sigma)U_{\infty}(\sigma)(\sigma\overline{E}-\overline{A})V_{\infty}(\sigma)\hat{T}_R(\sigma) = diag\left[\sigma I_{r(q_1-1)}, \frac{1}{\sigma^{q_1-1}}S^{\infty}_{A(\sigma)}(\sigma)\right] \tag{3.6}$$

It is obvious that equation (3.6) gives the Smith-McMillan form $S^{\infty}_{\sigma\overline{E}-\overline{A}}(\sigma)$ of the pencil $\sigma\overline{E}-\overline{A}$ at $\sigma = \infty$, i.e.

$$S^{\infty}_{\sigma\overline{E}-\overline{A}}(\sigma) = diag\left[\sigma I_{r(q_1-1)}, \frac{1}{\sigma^{q_1-1}}S^{\infty}_{A(\sigma)}(\sigma)\right] \tag{3.7}$$

The IEDs of $\sigma\overline{E}-\overline{A}$ are the FEDs $\sigma^{\overline{\mu}_j}$ of the dual pencil $\sigma\left(\sigma^{-1}\overline{E}-\overline{A}\right)=\overline{E}-\sigma\overline{A}$ at $\sigma=0$. From (3.7) the local Smith form $S^0_{\overline{E}-\sigma\overline{A}}(\sigma)$ at $\sigma=0$ of the dual pencil $\overline{E}-\sigma\overline{A}$ is given by

$$\begin{aligned} S^0_{\overline{E}-\sigma\overline{A}}(\sigma) &= \sigma S^{\infty}_{\sigma\overline{E}-\overline{A}}(\sigma^{-1}) \overset{(3.7)}{=} \sigma\ diag\left[\frac{1}{\sigma}I_{r(q_1-1)},\sigma^{q_1-1}S^{\infty}_{A(\sigma)}\left(\frac{1}{\sigma}\right)\right] \\ &\overset{(3.5)}{=} diag\left[I_{r(q_1-1)},\sigma^{q_1}diag\left[\underbrace{\frac{1}{\sigma^{q_1}},\ldots,\frac{1}{\sigma^{q_k}},I_{v-k}}_{v},\underbrace{\sigma^{\widehat{q}_{v+1}},\ldots,\sigma^{\widehat{q}_r}}_{r-v}\right]\right] \\ &= diag\left[I_{r(q_1-1)+1},\sigma^{q_1-q_2},...,\sigma^{q_1-q_k},I_{v-k}\sigma^{q_1},\underbrace{\sigma^{q_1+\widehat{q}_{v+1}},\ldots,\sigma^{q_1+\widehat{q}_r}}_{r-v}\right] \end{aligned}$$

so the orders $\overline{\mu}_j$ of the *IEDs* $\sigma^{\overline{\mu}_j}$ of $\sigma\overline{E}-\overline{A}$ are given by

$$\begin{aligned} \overline{\mu}_j &= 0, \quad j=1,2,...,r(q_1-1)+1 \\ \overline{\mu}_j &= q_1-q_j \qquad j=r(q_1-1)+2,\ldots,r(q_1-1)+k \\ \overline{\mu}_j &= q_1 \qquad j=r(q_1-1)+k+1,\ldots,r(q_1-1)+v \\ \overline{\mu}_j &= q_1+\widehat{q}_j \qquad j=r(q_1-1)+v+1,\ldots,r(q_1-1)+r \end{aligned} \tag{3.8}$$

which obviously coincide with the multiplicities of non-trivial IED's of $A(\sigma)$. ■

Motivated by the above result we propose a generalization of the notion of *strict equivalence* of matrix pencils [8] to the case of general nonsingular polynomial matrices.

Definition 3.4. *(strict equivalence of polynomial matrices). Two nonsingular polynomial matrices* $A_i(\sigma)=A_{iq^i}\sigma^{q^i}+A_{iq^i-1}\sigma^{q^i-1}+\ldots+A_{i0}\in\mathbb{R}_p[\sigma]$, $i=1,2$ *are called strictly equivalent (SE) iff the block companion matrix pencils* $\sigma\overline{E}_i-\overline{A}_i\in\mathbb{R}[\sigma]^{p\times p}$ $i=1,2$ *associated to* $A_i(\sigma)$ *are strictly equivalent [8], i.e. iff there exist nonsingular* $M\in\mathbb{R}^{p\times p},Q\in\mathbb{R}^{p\times p}$ *such that*

$$\left[\sigma\overline{E}_1-\overline{A}_1\right]=M\left[\sigma\overline{E}_2-\overline{A}_2\right]Q \tag{3.9}$$

Notice that if the polynomial matrices $A_i(\sigma)$ $i=1,2$ in Definition 3.4 are pencils i.e. if $q^1=q^2=1$ so that $A_i(\sigma)=A_{i1}\sigma+A_{i0}\equiv\sigma\overline{E_i}-\overline{A}_i\in\mathbb{R}[\sigma]^{r_i\times r_i}\subseteq\mathbb{R}_p[\sigma]$, $i=1,2$ and $p=r_1=r_2$, then our notion of strict equivalence coincides with the classical one in [8]. From the fact that any nonsingular $A(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ as in (2.1) is SE to the pencil $\sigma\overline{E}-\overline{A}\in\mathbb{R}[\sigma]^{rq\times rq}$ as in (3.1) it follows easily that SE of polynomial matrices is an *equivalence relation* on $\mathbb{R}_p[\sigma]\times\mathbb{R}_p[\sigma]$. If $A_i(\sigma)\in\mathbb{R}_p[\sigma]$, $i=1,2$ are SE then we denote this fact by writting $A_1(\sigma)\overset{SE}{\sim}A_2(\sigma)$. We can now generalize a classical result regarding strictly equivalent matrix pencils [8].

Theorem 3.5. $A_1(\sigma)\overset{SE}{\sim}A_2(\sigma)$ *iff* $A_1(\sigma)\in\mathbb{R}_p[\sigma]$ *and* $A_2(\sigma)\in\mathbb{R}_p[\sigma]$ *have the same FEDs and IEDs.*

Proof. ($\Rightarrow$) i.e. $A_1(\sigma)\overset{SE}{\sim}A_2(\sigma)\Rightarrow A_1(\sigma)$ and $A_2(\sigma)$ have the same FEDs and IEDs. It is enough to combine definition 3.4 with the result of Lemma 3.3, in order to see that the matrices $A_1(\sigma), A_2(\sigma), \left[\sigma\overline{E}_1 - \overline{A}_1\right], \left[\sigma\overline{E}_2 - \overline{A}_2\right]$ share common FED's and IED's structure.

($\Leftarrow$) i.e. $A_1(\sigma)$ and $A_2(\sigma)$ have the same FEDs and IEDs $\Rightarrow A_1(\sigma) \overset{SE}{\sim} A_2(\sigma)$. In view of Lemma 3.3 we can obtain first order polynomial matrices $\left[\sigma\overline{E}_i - \overline{A}_i\right]$ such that $\left[\sigma\overline{E}_i - \overline{A}_i\right]$ and $A_i(\sigma)$ share common FED's and IED's structure. According to our assumption $A_1(\sigma)$, $A_2(\sigma)$ have the same FED's and IED's structure thus the same will hold for $\left[\sigma\overline{E}_1 - \overline{A}_1\right], \left[\sigma\overline{E}_2 - \overline{A}_2\right]$. Hence $\left[\sigma\overline{E}_1 - \overline{A}_1\right] \overset{SE}{\sim} \left[\sigma\overline{E}_2 - \overline{A}_2\right]$, which coincides with the definition of strict equivalence for $A_1(\sigma)$, $A_2(\sigma)$. ■

Theorem 3.5 states that the map

$$\begin{aligned} f \quad &: \quad \mathbb{R}_p[\sigma] \to \underbrace{\mathbb{R}[\sigma] \times \mathbb{R}[\sigma] \times \cdots \times \mathbb{R}[\sigma]}_{n+\mu \text{ times}}, \\ A(\sigma) \quad &\mapsto \quad fA(\sigma) = \{\text{set of FEDs and IEDs of } A(\sigma)\} \end{aligned}$$

is a *complete invariant* for SE, i.e. that $A_1(\sigma) \overset{SE}{\sim} A_2(\sigma) \Leftrightarrow fA_1(\sigma) = fA_1(\sigma)$. Also since for any nonsingular $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ as in (2.1) and $\sigma\overline{E} - \overline{A} \in \mathbb{R}[\sigma]^{rq\times rq}$ as in (3.1) we have that

$$A(s) \overset{SE}{\sim} \sigma\overline{E} - \overline{A} \overset{SE}{\sim} \begin{bmatrix} \sigma I_n - J_F & 0_{n,\mu} \\ 0_{\mu,n} & \sigma J_\infty - I_\mu \end{bmatrix} =: \sigma E - A$$

we see that the map

$$w : \mathbb{R}_p[\sigma] \to \mathbb{R}_p[\sigma] \qquad wA_i(\sigma) = \begin{bmatrix} \sigma I_n - J_F & 0_{n,\mu} \\ 0_{\mu,n} & \sigma J_\infty - I_\mu \end{bmatrix} \quad i = 1, 2 \tag{3.10}$$

is a *canonical map* for SE on $\mathbb{R}_p[\sigma]$ and that the Weierstrass form $\sigma E - A$ of $\sigma\overline{E} - \overline{A}$ is a *canonical form* on $\mathbb{R}_p[\sigma]$ in the sense that if we consider two SE polynomial matrices $A_i(\sigma) \in \mathbb{R}_p[\sigma], i = 1, 2$ and their respective SE pencils $\sigma\overline{E}_i - \overline{A}_i \in \mathbb{R}_p[\sigma]$ $i = 1, 2$ as in definition 3.4 and let $M_i \in \mathbb{R}^{p\times p}, Q_i \in \mathbb{R}^{p\times p}, i = 1, 2$ be the matrices transforming each pencil $\sigma\overline{E}_i - \overline{A}_i$, $i = 1, 2$ to their (common) Weierstrass form:

$$\sigma E - A = \begin{bmatrix} \sigma I_n - J_F & 0_{n,\mu} \\ 0_{\mu,n} & \sigma J_\infty - I_\mu \end{bmatrix} = M_1\left[\sigma\overline{E}_1 - \overline{A}_1\right]Q_1 = M_2\left[\sigma\overline{E}_2 - \overline{A}_2\right]Q_2 \tag{3.11}$$

so that

$$A_1(s) \overset{SE}{\sim} \left[\sigma\overline{E}_1 - \overline{A}_1\right] \overset{SE}{\sim} \begin{bmatrix} \sigma I_n - J_F & 0_{n,\mu} \\ 0_{\mu,n} & \sigma J_\infty - I_\mu \end{bmatrix} \overset{SE}{\sim} \left[\sigma\overline{E}_2 - \overline{A}_2\right] \overset{SE}{\sim} A_2(s) \tag{3.12}$$

then

$$A_1(\sigma) \overset{SE}{\sim} A_2(\sigma) \Leftrightarrow wA_1(\sigma) = wA_2(\sigma) = \begin{bmatrix} \sigma I_n - J_F & 0_{n,\mu} \\ 0_{\mu,n} & \sigma J_\infty - I_\mu \end{bmatrix} \in \mathbb{R}_p[\sigma] \tag{3.13}$$

4. Fundamental equivalence of DTARR's

Consider now the $q-th$ order Discrete Time Auto–Regressive Representation (DTARR)

$$A_q\xi_{k+q} + A_{q-1}\xi_{k+q-1} + ... + A_0\xi_k = 0 \tag{4.1}$$

We are interested in the behaviour (see bellow) of (4.1) over a specified finite time interval $k = 0, 1, ..., N$ where $N \in \mathbb{Z}^+$ is arbitrary subject to $N \geq q$. If σ denotes the forward shift operator $\sigma^i \xi_k = \xi_{k+i}$ then (4.1) can be written as

$$A(\sigma)\xi_k = 0,\ k = 0, 1, 2, ..., N - q \geq 0 \tag{4.2}$$

where $A(\sigma)$ as in (2.1) and $\xi_k \in \mathbb{R}^r, k = 0, 1, ..., N$ is a vector sequence. Notice that as the matrix $A_q \in \mathbb{R}^{r\times r}$ is not in general invertible (4.1) can not be solved by iterating forward, i.e. given the *initial conditions* $\xi_0, \xi_1, ..., \xi_{q-1}$ determine successively $\xi_q, \xi_{q+1}, ...$

The solution space or behaviour $\mathcal{B}^{(N)}_{A(\sigma)}$ of the DTARR (4.2) over the finite time interval $[0, N]$ is defined as

$$\mathcal{B}^{(N)}_{A(\sigma)} = \{\xi := (\xi_k)_{k=0,1,...,N} \subseteq \mathbb{R}^r \mid \xi_k \text{ satisfies (4.2) for } k \in [0, N]\} \subseteq (\mathbb{R}^r)^{N+1} \tag{4.3}$$

The following result which appears in [6], [16], characterizes the behaviour of a DTARR in terms of the finite and infinite Jordan pairs of the polynomial matrix $A(\sigma)$ associated with (4.2) and can be considered as a direct generalization of corresponding results for descriptor systems that appear in [3], [4], [5], [1], [2] and [7].

Theorem 4.1. *[6][16] The behaviour $\mathcal{B}^{(N)}_{A(\sigma)}$ of the DTARR (4.2) over the finite time interval $k = 0, 1, ..., N \geq q$ is given by*

$$\begin{aligned}\mathcal{B}^{(N)}_{A(\sigma)} \;=\; & \{(\xi_k)_{k=0,1,...,N} \subseteq \mathbb{R}^r \mid \exists\ a \in \mathbb{R}^n, b \in \mathbb{R}^\mu\ \forall\ k = 0, 1, ..., N \geq q : \\ & \xi_k = \begin{bmatrix} C_F J_F^k & C_\infty J_\infty^{N-k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}\Bigg\}\end{aligned}$$

and

$$\dim \mathcal{B}^{(N)}_{A(\sigma)} = rq = n + \mu \tag{4.4}$$

Notice that $\mathcal{B}^{(N)}_{A(\sigma)}$ depends on the *finite and infinite Jordan pairs* $(C_F \in \mathbb{R}^{r\times n},\ J_F \in \mathbb{R}^{n\times n})$ and $(C_\infty \in \mathbb{R}^{r\times \mu}, J_\infty \in \mathbb{R}^{\mu\times\mu})$ of $A(\sigma)$ and thus from (2.2),(2.3) and (2.10) on the (non trivial) FEDs and IEDs: $(\sigma - \lambda_i)^{m_{ij}}$ and $\sigma^{\mu_j}, i = 1, 2, \ldots, l, j = 1, 2, ..., r$ of $A(\sigma)$.

Theorem 4.2. *[6][16] Given the initial and the final conditions vectors*

$$\begin{aligned} x_0 \;&:= \begin{bmatrix} \xi_0^\top & \xi_1^\top & \cdots & \xi_{q-2}^\top & \xi_{q-1}^\top \end{bmatrix}^\top \in \mathbb{R}^{rq} \\ x_{N-(q-1)} \;&:= \begin{bmatrix} \xi_{N-(q-1)}^\top & \xi_{N-(q-2)}^\top & \cdots & \xi_{N-1}^\top & \xi_N^\top \end{bmatrix}^\top \in \mathbb{R}^{rq} \end{aligned} \tag{4.5}$$

then (4.2) has the unique solution

$$\xi_k = \begin{bmatrix} C_F J_F^k M_F & C_\infty J_\infty^{N-k} M_\infty \end{bmatrix} \begin{bmatrix} x_0 \\ x_{N-(q-1)} \end{bmatrix}, k = 0, 1, 2, ..., N \geq q \tag{4.6}$$

iff x_0 and $x_{N-(q-1)}$ satisfy the compatibility boundary conditions

$$\begin{bmatrix} x_0 \\ x_{N-(q-1)} \end{bmatrix} \in \ker \begin{bmatrix} J_F^{N-(q-1)} M_F & -M_F \\ -M_\infty & J_\infty^{N-(q-1)} M_\infty \end{bmatrix} \tag{4.7}$$

where $M_F \in \mathbb{R}^{n\times rq}$, $M_\infty \in \mathbb{R}^{\mu\times rq}$ *are defined by*

$$\begin{bmatrix} M_F \\ M_\infty \end{bmatrix} := \begin{bmatrix} C_F & C_\infty J_\infty^{q-1} \\ C_F J_F & C_\infty J_\infty^{q-2} \\ \vdots & \vdots \\ C_F J_F^{q-1} & C_\infty \end{bmatrix}^{-1} \in \mathbb{R}^{rq\times rq} \tag{4.8}$$

(the fact that the matrix inside the brackets in the right hand side of (4.8) is non-singular is proved in [9] (Theorem 7.3 page 189).

Let $(\xi_k)_{k=0,1,\dots,N} \in \mathcal{B}^{(N)}_{A(\sigma)}$ be a solution of (4.2) and define the vectors

$$x_k := \begin{bmatrix} \xi_k \\ \xi_{k+1} \\ \vdots \\ \xi_{k+q-1} \end{bmatrix} = \begin{bmatrix} I_r \\ I_r\sigma \\ \vdots \\ I_r\sigma^{q-1} \end{bmatrix} \xi_k \in \mathbb{R}^{rq}, \quad k = 0,1,2,...,N \tag{4.9}$$

where we assign the values $\xi_k = 0$ for $k > N$. In view of (4.2) and the form $\sigma\overline{E} - \overline{A}$ in (3.1) the sequence $(x_k)_{k=0,1,\dots N} \in \mathcal{B}^{(N)}_{\sigma\overline{E}-\overline{A}}$ satisfies the DTARR

$$\left(\sigma\overline{E} - \overline{A}\right) x_k = 0, \quad k = 0,1,2,...,N-1 \tag{4.10}$$

where

$$\begin{aligned} \mathcal{B}^{(N)}_{\sigma\overline{E}-\overline{A}} \;=\; & \Big\{ x := (x_k)_{k=0,1,\dots,N} \subseteq \mathbb{R}^{rq}, \quad x_k \in \mathbb{R}^{rq} \\ & \mid x_k = [\overline{C}_F \overline{J}_F^k M_F, \overline{C}_\infty \overline{J}_\infty^{N-k-(q-1)} M_\infty] \begin{bmatrix} x_0 \\ x_{N-(q-1)} \end{bmatrix}, \quad k = 0,1,2,...,N \Big\} \end{aligned}$$

and from Theorem 4.1

$$\dim \mathcal{B}_{\sigma\overline{E}-\overline{A}} = (rq)\cdot 1 = n + \mu$$

where $\overline{C}_F \in \mathbb{R}^{rq\times n}, \overline{J}_F \equiv J_F \in \mathbb{R}^{n\times n}$ a finite Jordan pair and $\overline{C}_\infty \in \mathbb{R}^{rq\times\mu}, \overline{J}_\infty \equiv J_\infty \in \mathbb{R}^{\mu\times\mu}$ an infinite Jordan pair respectively of $\left[\sigma\overline{E} - \overline{A}\right] \in \mathbb{R}[\sigma]^{rq\times rq}$, and it can easily be proved that

$$[\overline{C}_F, \overline{C}_\infty] = \begin{bmatrix} C_F & C_\infty J_\infty^{q-1} \\ C_F J_F & C_\infty J_\infty^{q-2} \\ \vdots & \vdots \\ C_F J_F^{q-1} & C_\infty \end{bmatrix}$$

$M_F \in \mathbb{R}^{n\times rq}, M_\infty \in \mathbb{R}^{\mu\times rq}$ and defined through $\begin{bmatrix} M_F \\ M_\infty \end{bmatrix} := [\overline{C}_F, \overline{C}_\infty]^{-1} \in \mathbb{R}^{rq\times rq}$ and where the *initial and final conditions vectors* $x_0 \in \mathbb{R}^{rq}, x_{N-(q-1)} \in \mathbb{R}^{rq}$ satisfy the compatibility boundary condition (4.7).

A first observation on the dimensions of the behaviours of (4.2) and (4.10), shows there must be bijective maps between $\mathcal{B}^{(N)}_{A(\sigma)}$ and $\mathcal{B}^{(N)}_{\sigma\overline{E}-\overline{A}}$. These maps are given in the following

Proposition 4.3. *The polynomial maps*

$$S_{rq}(\sigma) : \mathcal{B}^{(N)}_{A(\sigma)} \to \mathcal{B}^{(N)}_{s\overline{E}-\overline{A}} \mid \xi_k \mapsto x_k = S_{rq}(\sigma)\xi_k,\ S_{rq}(\sigma) = \begin{bmatrix} I_r \\ I_r\sigma \\ \vdots \\ I_r\sigma^{q-1} \end{bmatrix} \in \mathbb{R}[\sigma]^{rq\times r} \tag{4.11}$$

and

$$L_{rq}(\sigma) : \mathcal{B}^{(N)}_{s\overline{E}-\overline{A}} \to \mathcal{B}^{(N)}_{A(\sigma)} \mid x_k \mapsto \xi_k = L_{rq}(\sigma)\, x_k, L_{rq}(\sigma) = \begin{bmatrix} I_r & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{r\times rq} \tag{4.12}$$

are bijective.

Proof. It is enough to notice that every solution of (4.2) is mapped through (4.11) to a solution of (4.10) while every solution of (4.10) is mapped through (4.12) to a solution of (4.2). ■

The question of existence of bijective polynomial maps between the behaviours of generic DTARRs of the form (4.2), is naturally arising in view of the existence of bijective polynomial maps relating the behaviour of a generic DTARR (4.2) to the behaviour of its first order representation (4.10). With this background we propose the following definition

Definition 4.4. *Two DTARRs*

$$A_i(\sigma)\,\xi^i_k = 0, \quad k = 0,1,2,...,N - q^i \geq 0, \qquad i = 1,2 \tag{4.13}$$

where $A_i(\sigma) = A_{iq^i}\sigma^{q^i} + A_{iq^i-1}\sigma^{q^i_1-1} + \ldots + A_{i1}\sigma + A_{i0} \in \mathbb{R}[\sigma]^{r_i\times r_i}$*,*$rank_{\mathbb{R}(\sigma)} A_i(\sigma) = r_i$*,* $i = 1,2$ *will be called fundamentally equivalent (FE) over the finite time interval* $k = 0,1,2,...N \geq \max\left\{q^1, q^2\right\}$ *iff there exists a bijective polynomial map* $Q_{12}(\sigma) = Q_{12v}\sigma^v + ... + Q_{121}\sigma + Q_{120} \in \mathbb{R}[\sigma]^{r_2\times r_1}$ *:*$\mathcal{B}^{(N)}_{A_1(\sigma)} \to \mathcal{B}^{(N)}_{A_2(\sigma)}$ *between their respective behaviors so that*

$$\xi^1 := (\xi^1_k)_{k=0,1,...,N} \in \mathcal{B}^{(N)}_{A_1(\sigma)} \mapsto \xi^2 := (\xi^2_k)_{k=0,1,...,N} \in \mathcal{B}^{(N)}_{A_2(\sigma)}$$

or

$$\xi^1_k \mapsto \xi^2_k = Q_{12}(\sigma)\,\xi^1_k, \quad k = 0,1,...,N$$

The above definition essentially suggests that two DTARRs will be termed FE, if their behaviours are isomorphic and particularly iff their behaviours are related through a bijective polynomial map. The following result is a direct consequence of the definition of FE.

Proposition 4.5. *The DTARR in (4.2) and (4.10) are fundamentally equivalent over the finite time interval* $k = 0,1,\ldots,N$*.*

Proof. This follows directly from Proposition 4.3 in view of the existence of the bijective polynomial maps $S_{rq}(\sigma) : \mathcal{B}^{(N)}_{A(\sigma)} \to \mathcal{B}^{(N)}_{s\overline{E}-\overline{A}}$ and $L_{rq}(\sigma) : \mathcal{B}^{(N)}_{s\overline{E}-\overline{A}} \to \mathcal{B}^{(N)}_{A(\sigma)}$ are given by (4.11) and (4.12). ■

Definition 4.6. *The DTARR (4.10) will be called a first order realization of the DTARR (4.2).*

From Definition 4.4 it follows that a necessary condition for the DTARRs in (4.13) to be fundamentally equivalent is that

$$\dim \mathcal{B}^{(N)}_{A_1(\sigma)} = r_1 q^1 = r_2 q^2 = \dim \mathcal{B}^{(N)}_{A_2(\sigma)} \tag{4.14}$$

In the following we show that a *necessary and sufficient* condition for the DTARRs in (4.13) to be FE is that $A_1(\sigma)\mathbb{R}[\sigma]^{r_1\times r_1}$ and $A_2(\sigma)\mathbb{R}[\sigma]^{r_2\times r_2}$ have the same FEDs and IEDs and that this n. & s. condition implies the necessary condition in (4.14)

The next proposition states that if the polynomial matrices $A_i(\sigma) \in \mathbb{R}[\sigma]^{r_i\times r_i}$, $i = 1, 2$ in Definition 4.4 are regular matrix pencils i.e. if $q^1 = q^2 = 1$, so that $A_i(\sigma) = A_{i1}\sigma + A_{i0} \equiv \sigma\overline{E}_i - \overline{A}_i \in \mathbb{R}[\sigma]^{r_i\times r_i}$, $i = 1, 2$ and $r_1 = r_2 = p$ then the notions of FE of the DTARR $\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, k = 0, 1, ..., N-1, i = 1, 2$ coincides with the notion *strict equivalence* of the matrix pencils $\sigma\overline{E}_i - \overline{A}_i \in \mathbb{R}[\sigma]^{p\times p}$, $i = 1, 2$.

Proposition 4.7. *Let* $\sigma\overline{E}_i - \overline{A}_i \in \mathbb{R}[\sigma]^{r_i\times r_i}, rank_{\mathbb{R}(\sigma)}\left[\sigma\overline{E}_i - \overline{A}_i\right] = r_i,\ i = 1, 2.$ *Then the DTAR representations* $\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, i = 1, 2, k = 0, 1, ..., N-1$ *are FE over the finite time interval* $k = 0, 1, ..., N$ *iff the matrix pencils* $\sigma\overline{E}_i - \overline{A}_i \in \mathbb{R}[\sigma]^{r_i\times r_i}$, $i = 1, 2$ *are SE.*

Proof. $(\Rightarrow)(\left[\sigma\overline{E}_1 - \overline{A}_2\right] \overset{SE}{\sim} \left[\sigma\overline{E}_1 - \overline{A}_2\right] \Rightarrow \left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, i = 1, 2, k = 0, 1, ..., N-1$ are FE.) $\left[\sigma\overline{E}_1 - \overline{A}_2\right] \overset{SE}{\sim} \left[\sigma\overline{E}_1 - \overline{A}_2\right] \Leftrightarrow r_1 = r_2 = p$ and $\exists$ nonsingular matrices $M \in \mathbb{R}^{p\times p}, Q \in \mathbb{R}^{p\times p} : \left[\sigma\overline{E}_1 - \overline{A}_1\right] = M\left[\sigma\overline{E}_2 - \overline{A}_2\right] Q$ from which by multiplying both sides by x^1_k we obtain that $x^2_k = Qx^1_k \in \mathcal{B}_{\sigma\overline{E}_2-\overline{A}_2}, k = 0, 1, 2, ..., N$. Since Q is square and invertible from $\dim \mathcal{B}^{(N)}_{\sigma\overline{E}_1-\overline{A}_1} = p = \dim \mathcal{B}^{(N)}_{\sigma\overline{E}_2-\overline{A}_2}$ it follows that

$$Q : \mathcal{B}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]} \to \mathcal{B}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]} \mid x^1_k \mapsto x^2_k = Qx^1_k, \quad k = 0, 1, 2, ..., N$$

is the bijection between the respective behaviors.

$(\Leftarrow)(\left[\sigma\overline{E}_i - \overline{A}_i\right] \in \mathbb{R}[\sigma]^{r_i\times r_i}, i = 1, 2$ are such that $\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, i = 1, 2, k = 0, 1, ..., N-1$ are FE $\Rightarrow \left[\sigma\overline{E}_1 - \overline{A}_2\right] \overset{SE}{\sim} \left[\sigma\overline{E}_1 - \overline{A}_2\right]$).

$\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, i = 1, 2, k = 0, 1, ..., N-1$ are FE $\overset{(4.14)}{\Rightarrow} \dim \mathcal{B}^{(N)}_{\sigma\overline{E}_1-\overline{A}_1} = r_1 = \dim \mathcal{B}^{(N)}_{\sigma\overline{E}_2-\overline{A}_2} = r_2 =: p$ and there exists a $p \times p$ bijective polynomial map $Q(\sigma) : \mathcal{B}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]} \to \mathcal{B}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]}$, $x^1_k \mapsto x^2_k$, $x^2_k = Q(\sigma)x^1_k, \quad k = 0, 1, 2, ..., N$. Let $Q(\sigma) = Q_v\sigma^v + ... + Q_1\sigma + Q_0 \in \mathbb{R}[\sigma]^{p\times p}$, be such a bijective polynomial map. Then $x^2_k = Q(\sigma)x^1_k = Q_v x^1_{k+v} + Q_{v-1}x^1_{k+v-1} + ... + Q_1 x^1_{k+1} + Q_0 x^1_k$, for $k = 0, 1, 2, ..., N$ can be written in matrix form as

$$\underbrace{\begin{bmatrix} x^2_0 \\ x^2_1 \\ \vdots \\ x^2_{N-1} \\ x^2_N \end{bmatrix}}_{p(N+1)\times 1} = \underbrace{\begin{bmatrix} Q_0 & Q_1 & ... & Q_{v-1} & Q_v & 0 & ... & 0 & 0 \\ 0 & Q_0 & Q_1 & ... & Q_{v-1} & Q_v & ... & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & Q_0 & Q_1 & ... & Q_{v-1} & Q_v & 0 \\ 0 & 0 & ... & 0 & Q_0 & Q_1 & ... & Q_{v-1} & Q_v \end{bmatrix}}_{p(N+1)\times p(N+v+1)} \underbrace{\begin{bmatrix} x^1_0 \\ x^1_1 \\ \vdots \\ x^1_{N+v-1} \\ x^1_{N+v} \end{bmatrix}}_{p(N+v+1)\times 1} \tag{4.15}$$

Let $L_N \in \mathbb{R}^{p(N+1)\times p(N+v+1)}$ the Toeplitz matrix in (4.15). Now due to the assumption that $Q(\sigma) : \mathcal{B}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]} \to \mathcal{B}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]}$ is bijective, $L_N : \mathbb{R}^{p(N+v+1)} \to \mathbb{R}^{p(N+1)}$ must also be bijective. In order L_N to be injective the number of columns of L_N should be equal or less than the number of its rows, thus $v = 0$ and $Q(\sigma) \equiv Q_0 \in \mathbb{R}^{p\times p}$ must be a constant invertible matrix. Hence $\left[\sigma\overline{E}_i - \overline{A}_i\right] x_k^i = 0, i = 1,2, k = 0,1,...,N-1$ are FE $\Rightarrow \exists$ nonsingular $Q_0 \in \mathbb{R}^{p\times p} : x_k^2 = Q_0 x_k^1, \quad k = 0,1,2,...,N$. Thus for every solution x_k^1 satisfying $\left[\sigma\overline{E}_1 - \overline{A}_1\right] x_k^1 = 0$, we have $\left[\sigma\overline{E}_2 - \overline{A}_2\right] x_k^2 = 0$ where $x_k^2 = Q_0 x_k^1$, which can be written as

$$\begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_{k+1}^1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_{k+1}^1 \end{bmatrix} = 0, \quad k = 0,1,...,N-1 \tag{4.16}$$

or equivalently

$$\ker \begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix} \subseteq \ker \begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix} \tag{4.17}$$

Inversely every solution x_k^2 satisfying $\left[\sigma\overline{E}_2 - \overline{A}_2\right] x_k^2 = 0$, can be written as $x_k^2 = Q_0 x_k^1$ where x_k^1 satisfies $\left[\sigma\overline{E}_1 - \overline{A}_1\right] x_k^1 = 0$. Thus

$$\begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_{k+1}^1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_{k+1}^1 \end{bmatrix} = 0, \quad k = 0,1,...,N-1 \tag{4.18}$$

$$\ker \begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix} \subseteq \ker \begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix} \tag{4.19}$$

Combining (4.17) and (4.19) we get $\ker \begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix} = \ker \begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix}$, which in implies the existence of a nonsingular $M_0 \in \mathbb{R}^{p\times p}$ s.t.

$$M_0 \begin{bmatrix} -\overline{A}_1 & \overline{E}_1 \end{bmatrix} = \begin{bmatrix} -\overline{A}_2 Q_0 & \overline{E}_2 Q_0 \end{bmatrix}$$

i.e. $\left[\sigma\overline{E}_1 - \overline{A}_2\right] \overset{SE}{\sim} \left[\sigma\overline{E}_1 - \overline{A}_2\right]$. ■

We can now state the main result of the paper which associates the notion of FE of two DTARRs to the notion of SE of polynomial matrices proposed in the previous section.

Theorem 4.8. *Let* $A_i(\sigma) \in \mathbb{R}[\sigma]^{r_i\times r_i}$, $rank_{\mathbb{R}(\sigma)} A_i(\sigma) = r_i, i = 1,2$. *Then the DTAR-representations*

$$A_i(\sigma)\,\xi_k^i = 0, \quad k = 0,1,2,...,N - q^i \geq 0, \qquad i = 1,2 \tag{4.20}$$

are fundamentally equivalent over the finite time interval $k = 0,1,...,N \geq \max\left\{q^1, q^2\right\}$ *iff the polynomial matrices* $A_i(\sigma)$, $i = 1,2$ *are strictly equivalent.*

Proof. $(\Rightarrow)$ $(A_1(\sigma) \overset{SE}{\sim} A_2(\sigma) \Rightarrow A_i(\sigma)\,\xi_k^i = 0, k = 0,1,2,...,N - q^i \geq 0, i = 1,2$ are FE)

Let $A_i(\sigma) = A_{iq^i}\sigma^{q^i} + A_{iq^i-1}\sigma^{q^i-1} + \ldots + A_{i1}\sigma + A_{i0} \in \mathbb{R}_p[\sigma]$, $i = 1,2$, $p = r_1 q^1 = r_2 q^2$ and assume that $A_1(\sigma) \overset{SE}{\sim} A_2(\sigma)$. Consider now the first order *realizations* of the DTARRs in (4.20) over the finite time interval $k = 0,1,2,\ldots,N$:

$$\left[\sigma\overline{E}_i - \overline{A}_i\right] x_k^i = 0, \quad k = 0,1,2,...,N-1, \quad i = 1,2 \tag{4.21}$$

Due to our assumption that $A_i(\sigma)$ SE, the matrix pencils $\left[\sigma\overline{E}_i - \overline{A}_i\right]$ will be also SE and hence according to the previous proposition (4.21) will be FE. Let $Q \in \mathbb{R}^{p\times p}$ be the bijective (constant) map, $Q : \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_1 - \overline{A}_1\right]} \to \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_2 - \overline{A}_2\right]} \mid x_k^1 \mapsto x_k^2 = Q x_k^1, \quad k = 0,1,2,...,N$, $S_{r_1 q^1}(\sigma) : \mathcal{B}^{(N)}_{A_1(\sigma)} \to \mathcal{B}^{(N)}_{s\overline{E}_1 - \overline{A}_1} \mid \xi_k^1 \mapsto x_k^1 = S_{r_1 q^1}(\sigma)\xi_k^1$ as defined in (4.11) and $L_{r_2q^2}(\sigma) : \mathcal{B}^{(N)}_{s\overline{E}_2-\overline{A}_2} \rightarrow \mathcal{B}^{(N)}_{A_2(\sigma)} \mid x^2_k \mapsto \xi^2_k = L_{r_2q^2}(\sigma)\, x^2_k$ as in (4.12). All three maps are bijective and it is easy to see that the mapping

$$Q_{12}(\sigma) = L_{r_2q^2}(\sigma)\, Q S_{r_1q^1}(\sigma)$$

is a bijective polynomial map (as a composition of bijective maps), mapping $\mathcal{B}^{(N)}_{A_1(\sigma)}$ to $\mathcal{B}^{(N)}_{A_2(\sigma)}$, which proves that (4.20) are FE. Following similar arguments we can determine a bijective polynomial map $Q_{21}(\sigma)$, that maps $\mathcal{B}^{(N)}_{A_2(\sigma)}$ to $\mathcal{B}^{(N)}_{A_1(\sigma)}$.

($\Rightarrow$) ($A_i(\sigma)\,\xi^i_k = 0, k = 0, 1, 2, ..., N - q^i \geq 0, i = 1, 2$ are FE$\Rightarrow A_1(\sigma) \overset{SE}{\sim} A_2(\sigma)$)

According to (4.5) the DTARRs $A_i(\sigma)\,\xi^i_k = 0$ and $\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0$, $i = 1, 2$ are fundamentally equivalent. Thus the following polynomial maps are bijective

$$L_{r_1q^1}(\sigma) : \mathcal{B}^{(N)}_{s\overline{E}_1-\overline{A}_1} \rightarrow \mathcal{B}^{(N)}_{A_1(\sigma)} \mid x^1_k \mapsto \xi^1_k = L_{r_1q^1}(\sigma) x^1_k$$

$$S_{r_2q^2}(\sigma) : \mathcal{B}^{(N)}_{A_2(\sigma)} \rightarrow \mathcal{B}^{(N)}_{s\overline{E}_2-\overline{A}_2} \mid \xi^2_k \mapsto x^2_k = S_{r_2q^2}(\sigma)\xi^2_k$$

By assumption there exists a polynomial bijective map

$$Q(\sigma) : \mathcal{B}^{(N)}_{A_1(\sigma)} \rightarrow \mathcal{B}^{(N)}_{A_2(\sigma)} \mid \xi^1_k \mapsto \xi^2_k = Q(\sigma)\xi^1_k$$

Therefore the composition $Q'(\sigma) := S_{r_2q^2}(\sigma)Q(\sigma)L_{r_1q^1}(\sigma)$

$$Q'(\sigma) : \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]} \rightarrow \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]} \mid x^1_k \mapsto x^2_k = Q'(\sigma)x^1_k$$

is a bijective polynomial map between $\mathcal{B}^{(N)}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]}, \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]}$. Thus the DTARRs $\left[\sigma\overline{E}_i - \overline{A}_i\right] x^i_k = 0, i = 1, 2$ are FE and by theorem 4.7 the pencils $\sigma\overline{E}_i - \overline{A}_i, i = 1, 2$ are SE which by definition 3.4 establishes the result. ■

The following diagram summarizes the bijective polynomial maps schema between the behaviours of (4.20) and their respective first order representations (4.21)

$$\begin{array}{ccc} \mathcal{B}^{(N)}_{A_1(\sigma)} & \overset{Q_{21}(\sigma)}{\longleftarrow} \atop \underset{Q_{12}(\sigma)}{\longrightarrow} & \mathcal{B}^{(N)}_{A_2(\sigma)} \\ L_{r_1q^1}(\sigma) \uparrow\downarrow S_{r_1q^1}(\sigma) & & S_{r_2q^2}(\sigma) \downarrow\uparrow L_{r_2q^2}(\sigma) \\ \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_1-\overline{A}_1\right]} & \overset{Q^{-1}}{\longleftarrow} \atop \underset{Q}{\longrightarrow} & \mathcal{B}^{(N)}_{\left[\sigma\overline{E}_2-\overline{A}_2\right]} \end{array} \tag{4.22}$$

We illustrate the above results via the following example

Example 4.9. *Consider the polynomial matrix* $A_1(\sigma) = \begin{bmatrix} \sigma^2 & 1 \\ 0 & \sigma^3 \end{bmatrix} \in \mathbb{R}[\sigma]^{2\times 2}$ *with* $r_1 = 2, q^1_1 = 3, n = 5$ *so that* $r_1q^1_1 = p = 6$, *and* $A_1(\sigma) \in \mathbb{R}_6[\sigma]$, $S^{\mathbb{C}}_{A_1(\sigma)} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma^5 \end{bmatrix}$. *The dual matrix of* $A_1(\sigma)$ *is* $\tilde{A}_1(\sigma) = \begin{bmatrix} \sigma & \sigma^3 \\ 0 & 1 \end{bmatrix}$ *and* $S^0_{\tilde{A}_1(\sigma)} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix}$, *so that*

$\mu_1^1 = 0$, $\mu_2^1 = 1$ and $\mu^1 = \mu_1^1 + \mu_2^1 = 1$. From $A_1(\sigma)$ by inspection

$$\sigma\overline{E}_1 - \overline{A}_1 = \begin{bmatrix} \sigma & 0 & -1 & 0 & 0 & 0 \\ 0 & \sigma & 0 & -1 & 0 & 0 \\ 0 & 0 & \sigma & 0 & -1 & 0 \\ 0 & 0 & 0 & \sigma & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{bmatrix} \in \mathbb{R}_6 [\sigma]$$

Consider now the polynomial matrix $A_2(\sigma) = \begin{bmatrix} \sigma^2 & 1 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma^2 \end{bmatrix} \in \mathbb{R}[\sigma]^{3\times 3}$ with $r_2 = 3$, $q_1^2 = 2, n = 5$ so that $r_2 q_1^2 = r_1 q_1^1 = p = 6$ and $A_2(\sigma) \in \mathbb{R}_6 [\sigma]$, $S^{\mathbb{C}}_{A_2(\sigma)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^5 \end{bmatrix}$. The dual matrix of $A_2(\sigma)$ is $\tilde{A}_2(\sigma) = \begin{bmatrix} 1 & \sigma^2 & 0 \\ 0 & \sigma & \sigma^2 \\ 0 & 0 & 1 \end{bmatrix}$ so that $S^0_{\tilde{A}_2(\sigma)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{bmatrix}$.and thus $\mu_1^2 = 0, \mu_2^2 = 0$, $\mu_3^2 = 1$and $\mu^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 = 1$. Obviously $A_1(\sigma), A_2(\sigma)$ share common finite and infinite elementary divisors and according to Theorem 3.5 they are *strictly equivalent*. From $A_2(\sigma)$ by inspection

$$\sigma\overline{E}_2 - \overline{A}_2 = \begin{bmatrix} \sigma & 0 & 0 & -1 & 0 & 0 \\ 0 & \sigma & 0 & 0 & -1 & 0 \\ 0 & 0 & \sigma & 0 & 0 & -1 \\ 0 & 1 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{bmatrix} \in \mathbb{R}_6 [\sigma]$$

Since $A_1(\sigma), A_2(\sigma)$ are strictly equivalent there exist constant non-singular matrices $M, Q \in \mathbb{R}^{6\times 6}$, such that $\sigma\overline{E}_1 - \overline{A}_1 = M(\sigma\overline{E}_2 - \overline{A}_2)Q$. It is easy to verify that two such matrices are

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

In view of diagram (4.22), Q is a (constant) bijection, mapping $\mathcal{B}^{(N)}_{[\sigma\overline{E}_1 - \overline{A}_1]}$ to $\mathcal{B}^{(N)}_{[\sigma\overline{E}_2 - \overline{A}_2]}$, while Q^{-1} is the inverse map. Furthermore by composing respectively, $L_{r_1 q^1}(\sigma), Q, S_{r_2 q^2(\sigma)}$ and $L_{r_2 q^2}(\sigma), Q^{-1}, S_{r_1 q^1(\sigma)}$, we obtain

$$Q_{12}(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sigma & 0 \\ 0 & \sigma \\ \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\sigma \end{bmatrix}$$

$$
Q_{21}(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

which are bijective polynomial maps between the behaviors $\mathcal{B}_{A_1(\sigma)}^{(N)}$ *and* $\mathcal{B}_{A_2(\sigma)}^{(N)}$ *for* $N \geq 3$*:*

$$
Q_{12}(\sigma) : \xi_k^1 \mapsto \xi_k^2 \qquad \begin{bmatrix} \xi_{1k}^2 \\ \xi_{2k}^2 \\ \xi_{3k}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} \xi_{1k}^1 \\ \xi_{2k}^1 \end{bmatrix} = \begin{bmatrix} \xi_{1k}^1 \\ \xi_{2k}^1 \\ -\xi_{2k+1}^1 \end{bmatrix}
$$

$$
Q_{21}(\sigma) : \xi_k^2 \mapsto \xi_k^1 \qquad \begin{bmatrix} \xi_{1k}^1 \\ \xi_{2k}^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_{1k}^2 \\ \xi_{2k}^2 \\ \xi_{3k}^2 \end{bmatrix} = \begin{bmatrix} \xi_{1k}^2 \\ \xi_{2k}^2 \end{bmatrix}
$$

5. Conclusions

We have introduced the notion of *fundamental equivalence* of DTARRs over a finite time interval and characterized the equivalence classes of polynomial matrices giving rise to fundamentally equivalent DTARRs. This new notion of equivalence is shown to be closely connected to the generalization, in the case of general polynomial matrices, of the notion of strict equivalence of matrix pencils, which was originally introduced in [8] .

The generalized version of strict equivalence for higher order polynomial matrices has the appealing, from an algebraic point of view property, of preserving the finite and infinite elementary divisors structure of polynomial matrices, a subject which has not been studied extensively in the past. This can probably be justified by the fact that the mainstream research has been focused on the study of continuous-time systems, where the structural invariants under preservation are the finite and infinite zeros of polynomial matrices (see [17], [20]). Furthermore, existing work on equivalence of first order continuous time systems (see [18], [21], [22], [27]), makes no distinction between the preservation of zeros and elementary divisors, since in the case of matrix pencils, the orders of IED's are related to the orders of zeros at infinity by the "plus one property" [13]. On the other hand finite and infinite elementary divisors play a crucial role on the behaviour of discrete time singular systems. When the higher than one order case is considered (see [6], [16]), the need for an equivalence relation that preserves both finite and infinite elementary divisors of polynomial matrices becomes apparent. Strict equivalence of polynomial matrices, is shown to be the necessary and sufficient condition for the existence of a bijective polynomial map between the behaviours of the associated DTARRs.

Further research on the subject could address issues like the existence of a closed formula to test strict equivalence of polynomial matrices (and hence FE of DTARR's), instead of characterizing SE polynomial matrices indirectly through their first order representations. The existence of such a closed formula condition, can be considered as an intermediate step towards the determination of a parametrization of all polynomial matrices that are SE to a given polynomial matrix and consequently characterize equivalence classes of DTARRs in the sense of FE.

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