On the fundamental matrix of the inverse of a polynomial matrix and applications

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***Abstract*— The aim of this work is twofold : a) it uses the fundamental matrix of the resolvent of a regular pencil in order to provide an algorithm for the computation of the fundamental matrix of the resolvent of a polynomial matrix, and b) it proposes a closed formula for the forward, backward and symmetric solution of an AutoRegressive Moving Average (ARMA). This closed formula is represented in terms of the fundamental matrix of the resolvent of one of the polynomial matrices that describes the ARMA representation.**

I. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{C}^{m\times n}$ the set of $m\times n$ matrices with elements in $\mathbb{C}$. If $E,A\in\mathbb{C}^{n\times n}$ and $zE-A$ is invertible for some $z\in\mathbb{C}$, then for some $R>0$ and $|z|>R$, we have the Laurent expansion at infinity of

$$\begin{gathered}(zE-A)^{-1}=\Phi_{-\mu}z^{\mu-1}+\cdots+\Phi_{-1}z^{0}+\\ +\Phi_{0}z^{-1}+\cdots+\Phi_{k}z^{-k-1}+\cdots=z^{-1}\sum_{i=-\mu}^{\infty}\Phi_{i}z^{-i}\end{gathered}$$

where the coefficient matrices $\Phi_i\in\mathbb{C}^{n\times n}$ are independent of z and are uniquely determined by E,A. $\Phi_i, i=-\mu,-\mu+1,..$ are known as the *(forward) fundamental matrix sequence* [4]. When $E=I_n$, the identity matrix, $(zI_n-A)^{-1}$, which always exists is often called the *resolvent* of A; thus $(zE-A)^{-1}$ was considered by [6] as a *generalized resolvent*. In [3] it is shown how to compute Φ_i given Φ_0 and Φ_{-1} and in [6] it is shown how to compute Φ_i from E,A using the Drazin inverse (where the Drazin inverse can be computed recursively as described in [9]). A number of important properties of the fundamental matrix sequence have been given in [3], [5]. The fundamental matrix sequence Φ_i plays a key role in the analysis of discrete singular systems, since the solution of discrete singular systems (forward, backward and symmetric) as well as many of the properties of these systems such as reachability, observability etc. can be described through the fundamental matrix sequence as has been shown in [1], [4] and [5].

Our first goal in this work is to show how to compute the generalized resolvent of a regular polynomial matrix $A(z)=A_0+A_1z+\cdots+A_qz^q, A_i\in\mathbb{R}^{n\times n}$ or otherwise how to compute the fundamental matrix sequence of the inverse of a regular polynomial matrix and thus extend the results presented by [3] and [6]. In order to accomplish this task we propose a matrix pencil $(zE-A)$ where the matrices E,A are determined in terms of the coefficient matrices A_i of the regular polynomial matrix $A(z)$ and then connect the fundamental matrix sequences of the regular polynomial matrix $A(z)$ and the ones of the matrix pencil $(zE-A)$. Our second goal is to propose an application of the fundamental matrix of a regular polynomial matrix to the solution of a discrete time AutoRegressive (ARMA) representation. More specifically we propose closed formulae for the forward solution (the solution in terms of the input sequence and the initial conditions), backward solution (the solution in terms of the final conditions and the input sequence) and the symmetric solution (the solution in terms of the boundary conditions (initial-final) conditions and the input sequence) of a discrete time ARMA representation. The whole theory is illustrated via examples.

II. Fundamental matrix and solutions of singular systems

Consider the singular dynamical system of equations

$$Ex_{k+1}=Ax_k+Bu_k\quad k=0,1,...,N-1 \tag{1}$$

with $x_k\in\mathbb{R}^n, u_k\in\mathbb{R}^m$. The interval of interest of index k is $[0,N]$, with u_k nonzero for $k=0,1,..,N$. By assuming that the pencil $zE-A$ is regular i.e. $\det(z_0E-A)\neq 0$ for some $z_0\in\mathbb{C}$, then for some $R>0$ and $|z|>R$, the Laurent series expansion about infinity for the resolvent matrix is given by

$$\begin{gathered}(zE-A)^{-1}=\Phi_{-\mu}z^{\mu-1}+\cdots+\Phi_{-1}z^{0}+\Phi_{0}z^{-1}+\\ +\cdots+\Phi_{k}z^{-k-1}+\cdots=z^{-1}\sum_{i=-\mu}^{\infty}\Phi_{i}z^{-i}\end{gathered} \tag{2}$$

where μ is the index of nilpotence and the sequence Φ_i is known as the (forward) fundamental matrix. Similarly for some $R>0$ and for $0<|z|<R$, the Laurent series expansion

about zero for the resolvent matrix is given by

$$(zE-A)^{-1} = V_p z^{-p} + \cdots + V_1 z^{-1} + \\ +V_0 z^0 + \cdots + V_{-k} z^k + \cdots = \sum_{i=-p}^{\infty} V_{-i} z^i \tag{3}$$

where the sequence V_{-i} is known as the (backward) fundamental matrix. The following properties of the fundamental matrix are well known [3] :

Theorem 1: With $(zE-A)$ regular and Φ_i defined by (2) :

1. $\Phi_i E - \Phi_{i-1} A = I\delta_i$
2. $E\Phi_i - A\Phi_{i-1} = I\delta_i$
3. $\Phi_i = \left\{ \begin{array}{cc} (\Phi_0 A)^i \Phi_0 & i \geq 0 \\ (-\Phi_{-1} E)^{-i-1} \Phi_{-1} & i < 0 \end{array} \right\}$
4. $\Phi_i E \Phi_j = \Phi_j E \Phi_i$
5. $\Phi_i E \Phi_j = \left\{ \begin{array}{cc} -\Phi_{i+j} & i<0, j<0 \\ \Phi_{i+j} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$
6. $\Phi_i A \Phi_j = \left\{ \begin{array}{cc} -\Phi_{i+j+1} & i<0, j<0 \\ \Phi_{i+j+1} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$

where δ_i is the Kronecker delta.

Explicit formulas for the coefficients Φ_i has been given in [1], [2], [4] and [6]. There have been several interpretations of the equation (1). >From a dynamical standpoint we may consider that the initial condition x_0 is given and that is desired to determine the state x_k in a forward fashion from the input sequence and the previous values of the semistate. We call this the *forward solution* of (1) and is given by [5] :

$$x_k = \Phi_k E x_0 + \sum\nolimits_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u_i \tag{4}$$

A variant of this is to consider x_N as given and then determine x_k in a backward fashion from the input and future values of the semistate. We call this the *backward solution* of (1) and is given by [5] :

$$x_k = -V_{k-N-1} E x_N + \sum\nolimits_{i=k-p}^{N-1} V_{k-i} B u_i \tag{5}$$

Another interpretation, arising in economics (where k might not be the time variable) and elsewhere, is to determine the semistate x_k for intermediate values of k, given the sequence $\{u_k\}$ and admissible x_0 and x_N. We call this the *symmetric solution* of (1) and is given by [5] :

$$x_k = \Phi_k E x_0 - \Phi_{-N+k} E x_N + \sum\nolimits_{i=0}^{N-1} \Phi_{k-i-1} B u_i \quad k = 1, 2, ..., N-1 \tag{6}$$

In the next section, using the properties of the fundamental matrix described above, we propose an algorithm for determining the fundamental matrix of the inverse of a regular polynomial matrix.

III. Fundamental matrix of the resolvent of a polynomial matrix

Consider the polynomial matrix

$$A(z) = A_0 + A_1 z + \ldots + A_q z^q \in R[z]^{r\times r}$$

By assuming that $A(z)$ is regular i.e. $\det A(z) \neq 0$, the Laurent series expansion about infinity for the resolvent matrix of $A(z)$ is defined by :

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r - 1} z^{\hat{q}_r - 1} + \ldots = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}$$

while the the Laurent series expansion about zero for the resolvent matrix of $A(z)$ is defined by :

$$A(z)^{-1} = N_\nu z^{-\nu} + \cdots + N_1 z^{-1} + \\ +N_0 z^0 + \cdots + N_{-k} z^k + \cdots = \sum_{i=-\nu}^{\infty} N_{-i} z^i$$

We introduce the following notation

$$H_k^{l,m} = \begin{bmatrix} H_k & H_{k-1} & \cdots & H_{k-m+1} \\ H_{k+1} & H_k & \cdots & H_{k-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{k+l-1} & H_{k+l-2} & \cdots & H_{k+l-m} \end{bmatrix} \in R^{lr\times mr}$$

where the subscript on $H_k^{l,m}$ indicates the subscript of the $(1,1)$ block-element of the matrix, whereas the superscript indicates its block dimensions. Denote also by

$$\tilde{E} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \in R^{qr\times qr} \tag{7}$$

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \in R^{qr\times qr}$$

In the following we will denote by Φ_i the coefficients of the Laurent series expansion about infinity of the inverse of the resolvent $\left(z\tilde{E}+\tilde{A}\right)^{-1}$ defined by:

$$\left(z\tilde{E}+\tilde{A}\right)^{-1} = \Phi_{-\mu} z^{\mu-1} + \cdots + \Phi_{-1} z^0 + \\ +\Phi_0 z^{-1} + \cdots + \Phi_k z^{-k-1} + \cdots = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i} \tag{8}$$

and by V_i the coefficients of the Laurent series expansion about zero, i.e.

$$\left(z\tilde{E}+\tilde{A}\right)^{-1} = V_p z^{-p} + \cdots + V_1 z^{-1} + \\ +V_0 z^0 + \cdots + V_{-k} z^k + \cdots = \sum_{i=-p}^{\infty} V_{-i} z^i$$

The next Theorem connects the coefficients of the Laurent expansion of $A(z)^{-1}$ and those of $\left(z\tilde{E}+\tilde{A}\right)^{-1}$.

Theorem 2: The coefficients $H_i\,(N_i)$ of the Laurent series expansion at infinity (zero) of $A(z)^{-1}$ and those $\Phi_i\,(V_i)$ of $\left(z\tilde{E}+\tilde{A}\right)^{-1}$ are connected by:

$$\Phi_i = H^{q,q}_{-q-qi} = \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}$$

$$V_i = N^{q,q}_{-qi} = \begin{bmatrix} N_{-qi} & N_{-qi-1} & \cdots & N_{-qi-q+1} \\ N_{-qi+1} & N_{-qi} & \cdots & N_{-qi-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qi+q-1} & N_{-qi+q-2} & \cdots & N_{-qi} \end{bmatrix}$$

Proof: Since H_i are coefficients of the Laurent series expansion of $A(z)^{-1}$ we have that

$$\begin{gathered} A\left(z\right)A\left(z\right)^{-1} = I_r \Leftrightarrow \\ \left(A_0 + A_1 z + \ldots + A_q z^q\right)\left(\sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}\right) = I_r \Leftrightarrow \\ \sum_{i=0}^{q} A_i H_{i-k} = \delta_k I_r \quad \left(\text{or } \sum_{i=0}^{q} H_{i-k} A_i = \delta_k I_r\right) \end{gathered} \tag{9}$$

Similarly we have that

$$\begin{gathered} A\left(z\right)A\left(z\right)^{-1} = I_r \Leftrightarrow \\ \left(A_0 + A_1 z + \ldots + A_q z^q\right)\left(\sum_{i=-\nu}^{\infty} N_{-i} z^{i}\right) = I_r \Leftrightarrow \\ \sum_{i=0}^{q} A_i N_{k-i} = \delta_k I_r \quad \left(\text{or } \sum_{i=0}^{q} N_{k-i} A_i = \delta_k I_r\right) \end{gathered} \tag{10}$$

Now, we can easily check that

$$\left(z\tilde{E}+\tilde{A}\right)\left(z^{-1}\sum_{i=-\mu}^{\infty}\Phi_i z^{-i}\right) = I_{qr}$$

or equivalently

$$\begin{gathered} \tilde{E}\Phi_i + \tilde{A}\Phi_{i-1} = \\ = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \times \\ \times \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix} + \\ + \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \\ \times \begin{bmatrix} H_{-qi} & H_{-qi-1} & \cdots & H_{-q-qi+1} \\ H_{-qi+1} & H_{-qi} & \cdots & H_{-q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi+q-1} & H_{-qi+q-2} & \cdots & H_{-qi} \end{bmatrix} \overset{(9)}{=} \\ = I_{qr}\delta_i \end{gathered}$$

Note that we have replaced the matrices E, A in (2) with $\tilde{E}, -\tilde{A}$ in (8). Similarly we can check that

$$\left(z\tilde{E}+\tilde{A}\right)\left(\sum_{i=-p}^{\infty} V_{-i} z^{i}\right) = I_{qr}$$

or equivalently

$$\begin{gathered} \tilde{E}V_{i+1} + \tilde{A}V_i = \\ = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \times \\ \times \begin{bmatrix} N_{-qi-q} & N_{-qi-q-1} & \cdots & N_{-qi-2q+1} \\ N_{-qi-q+1} & N_{-qi-q} & \cdots & N_{-qi-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qi-1} & N_{-qi-2} & \cdots & N_{-qi-q} \end{bmatrix} + \\ + \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \times \\ \times \begin{bmatrix} N_{-qi} & N_{-qi-1} & \cdots & N_{-qi-q+1} \\ N_{-qi+1} & N_{-qi} & \cdots & N_{-qi-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qi+q-1} & N_{-qi+q-2} & \cdots & N_{-qi} \end{bmatrix} \overset{(10)}{=} \\ = I_{qr}\delta_i \end{gathered}$$

Following the same reasoning, we can prove the second relation that concerns the coefficients of the Laurent expansion about zero of $A(z)^{-1}$ and $\left(z\tilde{E}+\tilde{A}\right)^{-1}$. ■

Based in the above Theorem it is now easy to compute the coefficients of the Laurent series expansion at infinity of $A(z)^{-1}$ by using the following algorithm.

Algorithm 1: Computation of the Laurent series expansion at infinity (zero) of $A(z)^{-1}$.

Step 1. Construct the matrices $\tilde{E}, \tilde{A}$ defined in (7).

Step 2. Determine the matrices $\Phi_0, \Phi_{-1}\,(V_1, V_0)$ of the resolvent $\left(z\tilde{E}+\tilde{A}\right)^{-1}$ by using one of the known computing techniques described in [1], [2], [4] and [6]. Compute the coefficients $H_{-2q+1}, \ldots, H_{q-1}$ $(N_{-2q+1}, \ldots, N_{q-1})$ from the following relations

$$\Phi_0 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix}$$

$$\Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} N_{-q} & N_{-q-1} & \cdots & N_{-2q+1} \\ N_{-q+1} & N_{-q} & \cdots & N_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-1} & N_{-2} & \cdots & N_{-q} \end{bmatrix}$$

$$V_0 = \begin{bmatrix} N_0 & N_{-1} & \cdots & N_{-q+1} \\ N_1 & N_0 & \cdots & N_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{q-1} & N_{q-2} & \cdots & N_0 \end{bmatrix}$$

Step 3. The rest terms can be determined by using the property (3) of Theorem 1

$$\Phi_i = \left\{ \begin{array}{cc} \left(-\Phi_0\tilde{A}\right)^i \Phi_0 & i \geq 0 \\ \left(-\Phi_{-1}\tilde{E}\right)^{-i-1} \Phi_{-1} & i < 0 \end{array} \right\} =$$

$$= \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}$$

Since Φ_i are the coefficients of the Laurent expansion at infinity of $\left(z\tilde{E} + \tilde{A}\right)^{-1}$, corresponding properties to the ones defined in Theorem 1 can now be established for polynomial matrices.

Theorem 3: With $A(z)$ regular and Φ_i defined by (8) :

1. $\Phi_i\tilde{E} + \Phi_{i-1}\tilde{A} = I\delta_i$
2. $\tilde{E}\Phi_i + \tilde{A}\Phi_{i-1} = I\delta_i$
3. $\Phi_i = \left\{ \begin{array}{cc} \left(-\Phi_0\tilde{A}\right)^i \Phi_0 & i \geq 0 \\ \left(-\Phi_{-1}\tilde{E}\right)^{-i-1} \Phi_{-1} & i < 0 \end{array} \right\}$
4. $\Phi_i\tilde{E}\Phi_j = \Phi_j\tilde{E}\Phi_i$
5. $\Phi_i\tilde{E}\Phi_j = \left\{ \begin{array}{cc} -\Phi_{i+j} & i < 0, j < 0 \\ \Phi_{i+j} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$
6. $\Phi_i\tilde{A}\Phi_j = \left\{ \begin{array}{cc} -\Phi_{i+j+1} & i < 0, j < 0 \\ \Phi_{i+j+1} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$

Example 1: Consider the inversion of the polynomial matrix

$$A(z) = \begin{bmatrix} z+1 & z-1 \\ 1 & z^2 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} z^2$$

Step 1. Construct the matrices $\tilde{E}, \tilde{A}$ defined in (7).

$$\tilde{E} = \begin{bmatrix} A_2 & A_1 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 2. Determine the matrices Φ_0, Φ_{-1} of the resolvent $\left(z\tilde{E} + \tilde{A}\right)^{-1}$ using the algorithm presented in [6]. Find c such that $\det\left(c\tilde{E} + \tilde{A}\right) \neq 0$ i.e. $c = 1$

$$\det\left[1 \times \tilde{E} + \tilde{A}\right] = 4$$

Then form the matrices

$$\hat{E} = \left(c\tilde{E} + \tilde{A}\right)^{-1}\tilde{E} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Then determine

$$\Phi_0 = \begin{bmatrix} H_{-2} & H_{-3} \\ H_{-1} & H_{-2} \end{bmatrix} = \hat{E}^D\left(c\tilde{E} + \tilde{A}\right)^{-1} =$$

$$= \begin{bmatrix} -1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{bmatrix} = -\hat{A}^D\left(I - \hat{E}\hat{E}^D\right)\left(c\tilde{E} + \tilde{A}\right)^{-1} =$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{E}^D$ denotes the Drazin inverse of $\tilde{E}$. Therefore

$$H_{-3} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \;;\; H_{-2} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$H_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \;;\; H_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \;;\; H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Step 3. The rest terms can be determined by using the property

(3) of Theorem 1

$$\Phi_i = \left\{ \begin{array}{ll} \left(-\Phi_0\tilde{A}\right)^i \Phi_0 \equiv 0_{4,4} & i \geq 0 \\ \left(-\Phi_{-1}\tilde{E}\right)^{-i-1} \Phi_{-1} & i < 0 \end{array} \right\} =$$

$$= \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}$$

IV. Applications to difference equations

Consider a linear, time invariant discrete time system, described by the difference equation:

$$A_0 y_k + A_1 y_{k+1} + \cdots + A_q y_{k+q} = B_0 u_k + \cdots + B_q u_{k+q}$$
$$k = 0, 1, ..., N - q$$

or otherwise

$$A(\sigma) y_k = B(\sigma) u_k \tag{11}$$

where σ denotes the shift-forward operator, $y_k : [0, N] \to \mathbb{R}^r$ is the output of the system, $u_k : [0, N] \to \mathbb{R}^m$ is a known input of the system, and

$$A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in \mathbb{R}[\sigma]^{r\times r}$$
$$B(\sigma) = B_0 + B_1\sigma + \cdots + B_q\sigma^q \in \mathbb{R}[\sigma]^{r\times m}$$

The above description is also known as the AutoRegressive Moving Average (ARMA) representation of our system. We may rewrite the above equations for $k = 0, 1, .., q - 1$ as follows

$$\tilde{E}x_{k+1} + \tilde{A}x_k = \tilde{B}v_k \quad k = 0, 1, ..., \left[\frac{N}{q}\right] - 1 \tag{12}$$

where

$$\tilde{E} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \in \mathbb{R}^{qr\times qr}$$

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{qr\times qr}$$

$$\tilde{B} = \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \in \mathbb{R}^{qr\times 2qr}$$

$$x_k = \begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix} \text{ and } v_k = \begin{bmatrix} u_{kq+2q-1} \\ u_{kq+2q-2} \\ \vdots \\ u_{kq+0} \end{bmatrix}$$

and $[N/q]$ denotes the integer part of the rational number N/q. Since N is usually not a multiple of the number q, we can always extend our interval $[0, N]$ to $[0, \hat{N}]$ where $\hat{N} = nq$, by adding new states or subtracting some of the last states of the system, always based on the initial-final conditions of the system and the solution of the system that will describe in the sequel. Therefore we assume in what follows, that $N \equiv \hat{N} = nq$.

Assume now that

$$\left(z\tilde{E} + \tilde{A}\right)^{-1} = z^{-1}\sum_{i=-\mu}^{\infty} \Phi_i z^{-i}$$

$$\left(z\tilde{E} + \tilde{A}\right)^{-1} = \sum_{i=-p}^{\infty} V_{-i} z^{i}$$

$$A(z)^{-1} = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}$$

$$A(z)^{-1} = \sum_{i=-\nu}^{\infty} N_{-i} z^{i}$$

Using the forward solution form described in (4), we can prove the following Theorem.

Theorem 4: The forward solution of (11) is the following:

$$y_k = \begin{bmatrix} H_{-k-q} & H_{-k-q+1} & \cdots & H_{-k-1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} +$$
$$+ \begin{bmatrix} H_{-k} & H_{-k+1} & \cdots & H_{\hat{q}_r} \end{bmatrix} \times$$
$$\times \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k+\hat{q}_r+q} \end{bmatrix} \tag{13}$$

Proof: Applying the form of the forward solution of singular systems described in (4) to the system (12) we have that

$$\begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix} = x_k = \Phi_k\tilde{E}x_0 + \sum\nolimits_{i=0}^{k+\mu-1} \Phi_{k-i-1}\tilde{B}v_i$$

$$= \begin{bmatrix} H_{-q-qk} & H_{-q-qk-1} & \cdots & H_{-2q-qk+1} \\ H_{-q-qk+1} & H_{-q-qk} & \cdots & H_{-2q-qk+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qk-1} & H_{-qk-2} & \cdots & H_{-qk-q} \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} +$$

$$+\sum_{i=0}^{k+\mu-1}\begin{bmatrix} H_{-qk+qi} & \cdots & H_{-qk+qi-q+1} \\ H_{-qk+qi+1} & \cdots & H_{-qk+qi-q+2} \\ \vdots & \ddots & \vdots \\ H_{-qk+qi+q-1} & \cdots & H_{-qk+qi} \end{bmatrix}\times$$

$$\times\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix}\begin{bmatrix} u_{iq+2q-1} \\ u_{iq+2q-2} \\ \vdots \\ u_{iq+0} \end{bmatrix}_i$$

$$=\begin{bmatrix} H_{-2q-qk+1} & \cdots & H_{-q-qk-1} & H_{-q-qk} \\ H_{-2q-qk+2} & \cdots & H_{-q-qk} & H_{-q-qk+1} \\ \vdots & \ddots & \vdots & \vdots \\ H_{-qk-q} & \cdots & H_{-qk-2} & H_{-qk-1} \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}+$$

$$+\begin{bmatrix} H_{-qk-q+1} & \cdots & H_{q\hat{q}_r-q} & H_{q\hat{q}_r-q+1} \\ H_{-qk-q+2} & \cdots & H_{q\hat{q}_r-q+1} & H_{q\hat{q}_r-q+2} \\ \vdots & \ddots & \vdots & \vdots \\ H_{-qk} & \cdots & H_{q\hat{q}_r-1} & H_{q\hat{q}_r} \end{bmatrix}\times$$

$$\times\begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix}\begin{bmatrix} u_0 \\ \vdots \\ u_{(k+\hat{q}_r+q)q-1} \\ u_{(k+\hat{q}_r+q)q} \end{bmatrix}$$

Taking the last of the above equations and replacing kq with k we get the forward formula (13). ■

A necessary and sufficient condition in order for the ARMA-representation (11) to have a solution, is that relation (13) is satisfied for $k=0,1,...,q-1$ i.e.

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}=\begin{bmatrix} H_{-q} & H_{-q+1} & \cdots & H_{-1} \\ H_{-q-1} & H_{-q} & \cdots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-2q+1} & H_{-2q+2} & \cdots & H_{-q} \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}+$$

$$+\begin{bmatrix} H_0 & \cdots & H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{-1} & \cdots & H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \end{bmatrix}\times$$

$$\times\begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{2q-1+\hat{q}_r} \end{bmatrix} \tag{14}$$

Using similar techniques and the backward solution form described in (4), we get the backward solution formula for the ARMA-representation (11).

Theorem 5: The backward solution of (11) is the following:

$$y_k=\begin{bmatrix} N_{N-k} & N_{N-k-1} & \cdots & N_{N-k-q+1} \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix}+$$

$$+\begin{bmatrix} N_{N-k-q} & N_{N-k-q-1} & \cdots & N_{-p} \end{bmatrix}\times \tag{15}$$

$$\times\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix}\begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{k-p} \end{bmatrix}$$

Proof: Let $\bar{N}=\left[\frac{N+1}{q}\right]-1$. We are interested for the vectors x_k where $0<k<\bar{N}=\left[\frac{N+1}{q}\right]-1\Rightarrow -qk+N+1-q>0$. Then using the relation (10) we obtain the following relation that we shall use in the sequel

$$-\begin{bmatrix} N_{-qk+N+1} & N_{-qk+N} & \cdots & N_{-qk+N-q+1} \\ N_{-qk+N+2} & N_{-qk+N+1} & \cdots & N_{-qk+N-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qk+N+q} & N_{-qk+N+q-1} & \cdots & N_{-qk+N+1} \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix}\overset{(10)}{=}$$

$$=\begin{bmatrix} N_{-qk+N+1-q} & N_{-qk+N-q} & \cdots & N_{-qk+N-2q+1} \\ N_{-qk+N+2-q} & N_{-qk+N+1-q} & \cdots & N_{-qk+N-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qk+N} & N_{-qk+N-1} & \cdots & N_{-qk+N+1-q} \end{bmatrix}\times \tag{16}$$

$$\times\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}$$

Applying the backward solution of singular systems described in (5) to the system (12) we have that

$$\begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix}=x_k=-V_{k-\bar{N}-1}\tilde{E}x_{\bar{N}}+\sum_{i=k-p}^{\bar{N}-1}V_{k-i}\tilde{B}v_i=$$

$$=-\begin{bmatrix} N_{-q(k-\bar{N}-1)} & \cdots & N_{-q(k-\bar{N}-1)-q+1} \\ N_{-q(k-\bar{N}-1)+1} & \cdots & N_{-q(k-\bar{N}-1)-q+2} \\ \vdots & \ddots & \vdots \\ N_{-q(k-\bar{N}-1)+q-1} & \cdots & N_{-q(k-\bar{N}-1)} \end{bmatrix}\times$$

$$
\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{\bar{N}q+q-1} \\ y_{\bar{N}q+q-2} \\ \vdots \\ y_{\bar{N}q+0} \end{bmatrix} +
$$

$$
+ \sum\nolimits_{i=k-p}^{\bar{N}-1} \begin{bmatrix} N_{-q(k-i)} & \cdots & N_{-q(k-i)-q+1} \\ N_{-q(k-i)+1} & \cdots & N_{-q(k-i)-q+2} \\ \vdots & \ddots & \vdots \\ N_{-q(k-i)+q-1} & \cdots & N_{-q(k-i)} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{iq+2q-1} \\ u_{iq+2q-2} \\ \vdots \\ u_{iq+0} \end{bmatrix}
$$

$$
\overset{\bar{N}=[\frac{N+1}{q}]-1}{=} - \begin{bmatrix} N_{-qk+N+1} & \cdots & N_{-qk+N-q+1} \\ N_{-qk+N+2} & \cdots & N_{-qk+N-q+2} \\ \vdots & \ddots & \vdots \\ N_{-qk+N+q} & \cdots & N_{-qk+N+1} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} N_{-qk+N+1} & \cdots & N_{-qp-q+1} \\ N_{-qk+N+2} & \cdots & N_{-qp-q+2} \\ \vdots & \ddots & \vdots \\ N_{-qk+N+q} & \cdots & N_{-qp} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{(k-p)q} \end{bmatrix}
$$

$$
\overset{(16)}{=} \begin{bmatrix} N_{-qk+N+1-q} & \cdots & N_{-qk+N-2q+1} \\ N_{-qk+N+2-q} & \cdots & N_{-qk+N-2q+2} \\ \vdots & \ddots & \vdots \\ N_{-qk+N} & \cdots & N_{-qk+N+1-q} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} N_{-qk+N+1} & \cdots & N_{-qp-q+1} \\ N_{-qk+N+2} & \cdots & N_{-qp-q+2} \\ \vdots & \ddots & \vdots \\ N_{-qk+N+q} & \cdots & N_{-qp} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{(k-p)q} \end{bmatrix}
$$

By taking the last of the above equations and replacing kq with k we get the backward formula (15). ■

A necessary and sufficient condition in order for the ARMA-representation (11) to have a solution, is that the relation (15) is satisfied $k = 0, 1, ..., q-1$ i.e.

$$
\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} = \begin{bmatrix} N_0 & N_{-1} & \cdots & N_{-q+1} \\ N_1 & N_0 & \cdots & N_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{q-1} & N_{q-2} & \cdots & N_0 \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} N_{-q} & \cdots & N_{-p} & 0 & \cdots & 0 \\ N_{-q+1} & \cdots & N_{-p+1} & N_{-p} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_{-1} & \cdots & N_{-p+q-1} & N_{-p+q-2} & \cdots & N_{-p} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{k-p} \end{bmatrix} \tag{17}
$$

Using the symmetric solution form described in (6), we get the symmetric solution formula for the ARMA-representation (11).

Theorem 6: The symmetric solution of (11) is the following:

$$
y_k = \begin{bmatrix} H_{-k-1} & H_{-k-2} & \cdots & H_{-k-q} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} +
$$

$$
+ \begin{bmatrix} H_{N-k} & H_{N-k-1} & \cdots & H_{N-q-k+1} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} H_{-q+N-k} & H_{-q+N-k-1} & \cdots & H_{-k} \end{bmatrix} \times \tag{18}
$$

$$
\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}
$$

Proof: Let $\bar{N}=\left[\frac{N+1}{q}\right]-1$. Using relation (10) we have

$$-\begin{bmatrix} H_{N-2q-qk+1} & \cdots & H_{N-3q-qk+2}\\ H_{N-2q-qk+2} & \cdots & H_{N-3q-qk+1}\\ \vdots & \ddots & \vdots \\ H_{N-q-qk} & \cdots & H_{N-2q-qk+1}\end{bmatrix}\times \begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix} = \begin{bmatrix} H_{N-q-qk+1} & \cdots & H_{N-2q-qk+2}\\ H_{N-q-qk+2}&\cdots&H_{N-2q-qk+1}\\ \vdots&\ddots&\vdots\\ H_{N-qk}&\cdots&H_{N-q-qk+1}\end{bmatrix}\times\begin{bmatrix}A_0&0&\cdots&0\\A_1&A_0&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ A_{q-1}&A_{q-2}&\cdots&A_0\end{bmatrix} \tag{19}$$

Applying the form of the symmetric solution of singular systems described in (6) to the system (12) we have that

$$\begin{bmatrix} y_{kq+q-1}\\ y_{kq+q-2}\\ \vdots\\ y_{kq+0}\end{bmatrix} = x_k = \Phi_k\tilde{E}x_0-\Phi_{-\bar{N}+k}\tilde{E}x_{\bar{N}}+ +\sum_{i=0}^{\bar{N}-1}\Phi_{k-i-1}\tilde{B}v_i$$

$$=\begin{bmatrix} H_{-q-qk} & \cdots & H_{-2q-qk+1}\\ H_{-q-qk+1} & \cdots & H_{-2q-qk+2}\\ \vdots & \ddots & \vdots\\ H_{-qk-1} & \cdots & H_{-qk-q}\end{bmatrix}\times\begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix}\begin{bmatrix}y_{q-1}\\ y_{q-2}\\ \vdots\\ y_0\end{bmatrix}-$$

$$-\begin{bmatrix} H_{-q+q\bar{N}-qk} & \cdots & H_{q\bar{N}-qk-2q-1}\\ H_{-q+q\bar{N}-qk+1} & \cdots & H_{q\bar{N}-qk-2q-2}\\ \vdots & \ddots & \vdots\\ H_{q\bar{N}-qk-1} & \cdots & H_{q\bar{N}-qk-q}\end{bmatrix}\times\begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix}\begin{bmatrix}y_{\bar{N}q+q-1}\\ y_{\bar{N}q+q-2}\\ \vdots\\ y_{\bar{N}q+0}\end{bmatrix}+$$

$$+\sum_{i=0}^{\bar{N}-1}\begin{bmatrix} H_{-qk+qi} & \cdots & H_{-qk+qi-q+1}\\ H_{-qk+qi+1} & \cdots & H_{-qk+qi-q+2}\\ \vdots & \ddots & \vdots\\ H_{-qk+qi+q-1} & \cdots & H_{-qk+qi}\end{bmatrix}\times\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0\\ 0 & B_q & \cdots & B_0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0\end{bmatrix}\begin{bmatrix}u_{iq+2q-1}\\ u_{iq+2q-2}\\ \vdots\\ u_{iq+0}\end{bmatrix}=$$

$$\stackrel{\bar{N}=\left[\frac{N+1}{q}\right]-1}{=}\begin{bmatrix} H_{-q-qk} & \cdots & H_{-2q-qk+1}\\ H_{-q-qk+1} & \cdots & H_{-2q-qk+2}\\ \vdots & \ddots & \vdots\\ H_{-qk-1} & \cdots & H_{-qk-q}\end{bmatrix}\times\begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix}\begin{bmatrix}y_{q-1}\\ y_{q-2}\\ \vdots\\ y_0\end{bmatrix}-$$

$$-\begin{bmatrix} H_{N-2q-qk+1} & \cdots & H_{N-3q-qk+2}\\ H_{N-2q-qk+2} & \cdots & H_{N-3q-qk+1}\\ \vdots & \ddots & \vdots \\ H_{N-q-qk} & \cdots & H_{N-2q-qk+1}\end{bmatrix}\times\begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix}\begin{bmatrix}y_N\\ y_{N-1}\\ \vdots\\ y_{N-q+1}\end{bmatrix}+$$

$$+\begin{bmatrix} H_{-2q+N-qk+1} & H_{-2q+N-qk} & \cdots & H_{-qk-q+1}\\ H_{-2q+N-qk+2} & H_{-2q+N-qk+1} & \cdots & H_{-qk-q+2}\\ \vdots & \vdots & \ddots & \vdots\\ H_{-q+N-qk} & H_{-q+N-qk-1} & \cdots & H_{-qk}\end{bmatrix}\times\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0\\ 0 & B_q & \cdots & B_0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0\end{bmatrix}\begin{bmatrix}u_N\\ u_{N-1}\\ \vdots\\ u_0\end{bmatrix}=$$

$$\stackrel{(19)}{=}\begin{bmatrix} H_{-q-qk} & \cdots & H_{-2q-qk+1}\\ H_{-q-qk+1} & \cdots & H_{-2q-qk+2}\\ \vdots & \ddots & \vdots\\ H_{-qk-1} & \cdots & H_{-qk-q}\end{bmatrix}\times\begin{bmatrix}A_q & A_{q-1} & \cdots & A_1\\ 0 & A_q & \cdots & A_2 \\ \vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&A_q\end{bmatrix}\begin{bmatrix}y_{q-1}\\ y_{q-2}\\ \vdots\\ y_0\end{bmatrix}+$$

$$+\begin{bmatrix} H_{N-q-qk+1} & \cdots & H_{N-2q-qk+2}\\ H_{N-q-qk+2}&\cdots&H_{N-2q-qk+1}\\ \vdots&\ddots&\vdots\\ H_{N-qk}&\cdots&H_{N-q-qk+1}\end{bmatrix}\times\begin{bmatrix}A_0&0&\cdots&0\\A_1&A_0&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ A_{q-1}&A_{q-2}&\cdots&A_0\end{bmatrix}\begin{bmatrix}y_N\\ y_{N-1}\\ \vdots\\ y_{N-q+1}\end{bmatrix}+$$

$$\begin{bmatrix} H_{-2q+N-qk+1} & \cdots & H_{-qk-q+1} \\ H_{-2q+N-qk+2} & \cdots & H_{-qk-q+2} \\ \vdots & \ddots & \vdots \\ H_{-q+N-qk} & \cdots & H_{-qk} \end{bmatrix} \times$$
$$\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Taking the last of the above equations and replacing kq with k we get the symmetric formula (18). ■

A necessary and sufficient condition in order for the ARMA-representation (11) to have a solution is that the relation (18) is satisfied for $k = 0, 1, ..., q-1$ and $k = N, N-1, ..., N-q+1$ i.e.

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} H_{-1} & H_{-2} & \cdots & H_{-q} \\ H_{-2} & H_{-3} & \cdots & H_{-q-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-q} & H_{-q-1} & \cdots & H_{-2q} \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} +$$
$$+ \begin{bmatrix} H_N & H_{N-1} & \cdots & H_{N-q+1} \\ H_{N-1} & H_{N-2} & \cdots & H_{N-q} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q+1} & H_{N-q} & \cdots & H_{N-2q+2} \end{bmatrix} \times \tag{20}$$
$$\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +$$
$$+ \begin{bmatrix} H_{N-q} & \cdots & H_0 \\ H_{N-q-1} & \cdots & H_{-1} \\ \vdots & \ddots & \vdots \\ H_{N-2q+1} & \cdots & H_{-q+1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

and

$$\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} = \begin{bmatrix} H_{-N-1} & \cdots & H_{-N-q} \\ H_{-N} & \cdots & H_{-N-q+1} \\ \vdots & \ddots & \vdots \\ H_{-N+q-2} & \cdots & H_{-N-1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} +$$
$$+ \begin{bmatrix} H_0 & \cdots & H_{-q+1} \\ H_1 & \cdots & H_{-q+2} \\ \vdots & \ddots & \vdots \\ H_{q-1} & \cdots & H_0 \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} + \tag{21}$$
$$+ \begin{bmatrix} H_{-q} & \cdots & H_{-N} \\ H_{-q+1} & \cdots & H_{-N+1} \\ \vdots & \ddots & \vdots \\ H_{-1} & \cdots & H_{-N+q-1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Other formulae for the forward, backward and symmetric solutions as well as the consistency of initial and/or final conditions of discrete time ARMA-representations can be found in [7]. An implementation of these formulae in the Maple symbolic language can be found in [8]. It is easily seen that the above results coincide with the ones presented in [5] when the $A(z) = zE - A$ or the known results from the state space theory when $A(z) = zI_n - A$.

Example 2: Consider the following discrete time ARMA representation:

$$\begin{bmatrix} z+1 & z-1 \\ 1 & z^2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{22}$$

Then from the first example we have that

$$\Phi_0 = \hat{E}^D \left(c\tilde{E} + \tilde{A}\right)^{-1} = \tag{23}$$
$$= \begin{bmatrix} -1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} H_{-2} & H_{-3} \\ H_{-1} & H_{-2} \end{bmatrix}$$

and

$$\Phi_{-1} = -\left(I_4 - \hat{E}\hat{E}^D\right) \hat{A}^D \left(c\tilde{E} + \tilde{A}\right)^{-1} = \tag{24}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{bmatrix}$$

Using property (3) of Theorem 3 we get that

$$\Phi_1 = \begin{bmatrix} H_{-4} & H_{-5} \\ H_{-3} & H_{-4} \end{bmatrix} = -\Phi_0 \tilde{A} \Phi_0 = \tag{25}$$

$$= \begin{bmatrix} -4 & -3 & 7 & 6 \\ 1 & 1 & -2 & -2 \\ 2 & 2 & -4 & -3 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Using (23, 24, 25) we conclude that

$$H_{-5} = \begin{bmatrix} 7 & 6 \\ -2 & -2 \end{bmatrix}; H_{-4} = \begin{bmatrix} -4 & -3 \\ 1 & 1 \end{bmatrix}$$

$$H_{-3} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}; H_{-2} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}; H_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The forward solution of the ARMA system is given by (13) and so

$$y_2 = \begin{bmatrix} H_{-4} & H_{-3} \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} H_{-2} & H_{-1} \end{bmatrix} \begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ 0 & B_0 & B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

which after some operations becomes

$$y_2 = \begin{bmatrix} -u_0 + u_1 + 2y_0 - y_0^2 + 2y_1^2 \\ -y_0 \end{bmatrix}$$

Following similar techniques, we get

$$y_3 = \begin{bmatrix} H_{-5} & H_{-4} \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} H_{-3} & H_{-2} & H_{-1} \end{bmatrix} \begin{bmatrix} B_0 & B_1 & B_2 & 0 & 0 \\ 0 & B_0 & B_1 & B_2 & 0 \\ 0 & 0 & B_0 & B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

which becomes

$$y_3 = \begin{bmatrix} 2u_0 - u_1 + u_2 - 4y_0 + 2y_0^2 - 3y_1^2 \\ -u_0 + y_0 - y_0^2 + y_1^2 \end{bmatrix}$$

According to (14), a necessary and sufficient condition in order for the ARMA-representation (22) to have a forward solution is

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} := \begin{bmatrix} y_0^1 \\ y_0^2 \\ y_1^1 \\ y_1^2 \end{bmatrix} = \begin{bmatrix} H_{-2} & H_{-1} \\ H_{-3} & H_{-2} \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 \\ H_{-1} & 0 \end{bmatrix} \begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ 0 & B_0 & B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} =$$

$$= \begin{bmatrix} y_0^1 \\ y_0^2 \\ u_0 - y_0^1 + y_0^2 - y_1^2 \\ y_1^2 \end{bmatrix}$$

i.e.

$$y_1^1 = u_0 - y_0^1 + y_0^2 - y_1^2$$

An algorithm for the implementation of the above forward formula for more steps can be found in [8].

V. Conclusions

We have shown that in order to compute the fundamental matrix sequence of the inverse of a regular polynomial matrix, it is enough to compute the fundamental matrix of the inverse of a matrix pencil, where the coefficients of the matrix pencil are written directly in terms of the matrix coefficients of the respective regular polynomial matrix. Closed formulae for the forward, backward and symmetric solution of discrete time ARMA representations has also been presented. The whole theory has been illustrated via examples. Further research is undergoing in order to provide tests for properties of discrete time ARMA-representations such as reachability and observability in terms of the fundamental matrix of the regular polynomial matrix that describes the system.

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