DFT calculation of the generalized and drazin inverse of a polynomial matrix

N. Karampetakis, S. Vologiannidis

Department of Mathematics

Aristotle University of Thessaloniki

Thessaloniki 54006, Greece

http://anadrasis.math.auth.gr

Objectives

- A new algorithm is presented for the determination of the *generalized inverse* and the *drazin inverse* of a polynomial matrix based on the discrete Fourier transform.
- The above algorithms are implemented in the Mathematica programming language.

Discrete Fourier Transform

Definition 1. In order for the finite sequence $X(k)$ and the sequence $\tilde{X}(k)$ to constitute a DFT pair the following relations should hold [Dudgeon, 1984]:

$$\tilde{X}(k) = \sum^{M} X(k)W^{-kr}, X(k) = \frac{1}{M+1}\sum^{M} \tilde{X}(k)W^{kr}$$

where $W = e^{\frac{2\pi j}{M+1}}$ and $X(k), \tilde{X}(k)$ are discrete argument matrix-valued functions, with dimensions $p \times m$.

Definition 2. In order for the finite sequence $X(k_1, k_2)$ and the sequence $\tilde{X}(r_1, r_2)$ to constitute a DFT pair the following relations should hold [Dudgeon, 1984]:

$$\tilde{X}(r_1, r_2) = \sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} X(k_1, k_2)W_1^{-k_1 r_1}W_2^{-k_2 r_2}, X(k_1, k_2) = \frac{1}{R}\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \tilde{X}(r_1, r_2)W_1^{k_1 r_1}W_2^{k_2 r_2}$$

where

$$W = e^{\frac{2\pi j}{M_i+1}}, R = (M_1+1)\times(M_2+1)$$

and $X(k_1, k_2), \tilde{X}(r_1, r_2)$ are discrete argument matrix-valued functions, with dimensions $p \times m$.

Generalized Inverse

For every matrix $A \in R^{p\times m}$, a unique matrix $A^{+} \in R^{m\times p}$, which is called generalized inverse, exists satisfying

(i) $AA^{+}A = A$

(ii) $A^{+}AA^{+} = A^{+}$

(iii) $\left(AA^{+}\right)^{T} = AA^{+}$

(iv) $\left(A^{+}A\right)^{T} = A^{+}A$

where A^{T} denotes the transpose of A. In the special case that the matrix A is square nonsingular matrix, the generalized inverse of A is simply its inverse i.e. $A^{+} = A^{-1}$.

In an analogous way we define the generalized inverse $A(s)^{+} \in R(s)^{m\times p}$ of the polynomial matrix $A(s) \in R[s]^{p\times m}$

Computation of the generalized inverse

[Karampetakis 1997] Let $A(s) \in R[s]^{p\times m}$ and

$$a(s,z) = \det\left[zI_p - A(s)A(s)^T\right] = a_0(s)z^p + a_1(s)z^{p-1} + \ldots + a_{p-1}(s)z + a_p(s),$$

$a_0(s) = 1$, be the characteristic polynomial of $A(s)A(s)^T$. Let $a_p(s) \equiv 0, \ldots, a_{k+1}(s) \equiv 0$ while $a_k(s) \neq 0$ and $\Lambda := \{s_i \in R : a_k(s_i) = 0\}$ Then the generalized inverse $A(s)^+$ of $A(s)$ for $s \in R - \Lambda$ is given by

$$A(s)^+ = -\tfrac{1}{a_k(s)} A(s)^T B_{k-1}(s),\; B_{k-1}(s) = a_0(s)\left(A(s)A(s)^T\right)^{k-1} + \ldots + a_{k-1}(s)I_p$$

If $k = 0$ is the largest integer such that $a_k(s) \neq 0$, then $A(s)^+ = 0$. For those $s_i \in \Lambda$ find the largest integer $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and then the generalized inverse $A(s_i)^+$ of $A(s_i)$ is given by

$$A(s_i)^+ = -\tfrac{1}{a_{k_i}(s_i)} A(s_i)^T B_{k_i-1}(s_i),\; B_{k_i-1}(s_i) = a_0(s)\left(A(s_i)A(s_i)^T\right)^{k-1} + \ldots + a_{k_i-1}(s_i)I_p$$

Computation of the generalized inverse via DFT

Step 1. (Evaluation of the polynomial a(s,z))

$$a(s,z)=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} a_{l_1,l_2} s^{l_1} z^{l_2} = \det\left[zI_p - A(s)A(s)^T \right]$$

- We use the following $R=(2pq+1)\times(p+1)$ interpolation points

  $u_i(r_j)=W_i^{-r_j}, i=1,2$ and $r_j=0,1,\dots,M_i$

  where $\mathrm{W}_i = e^{\frac{2\pi j}{M_i+1}} i=1,2; M_1=2pq; M_2=p$

- $$\tilde{a}_{r_1,r_2}=\det[u_2(r_2)I_p - A(u_1(r_1))A(u_1(r_1))^T]=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} a_{l_1,l_2} W_1^{-r_1 l_1} W_2^{-r_2 l_2}$$

  $[\tilde{a}_{r_1,r_2}]$ and $[a_{l_1,l_2}]$ form a DFT pair.

- $$a_{l_1,l_2}=\frac{1}{R}\sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2} \tilde{a}_{r_1,r_2} W_1^{r_1 l_1} W_2^{r_2 l_2}$$

Step 2. (Evaluate $a_k(s)$)

Find $k : a_{k+1}(s) = a_{k+2}(s) == a_p(s) = 0$ and $a_k(s) \neq 0$ or $a_{l_1,0} = a_{l_1,1} == a_{l_1,k+1} = 0 \ \forall l_1$ and $a_{l_1,k} \neq 0$ for some k.

Step 3. (Evaluate $B(s) = A(s)^T B_{k-1}(s)$ *where*

$$B_{k-1}(s) = a_0(s)\left(A(s)A(s)^T\right)^{k-1} + \ldots + a_{k-1}(s)I_p \text{)}$$

- We use the following $R = (2p-1)q+1$ interpolation points

$$u(r) = W^{-r}, W = e^{\frac{2\pi j}{(2p-1)q+1}}$$

- $$\tilde{B}_r = B(u(r)) = \sum_{l=0}^{n} B_l W^{-lr}$$

$[\tilde{B}_i]$ and $[B_l]$ form a DFT pair

- $$B_l = \frac{1}{R}\sum_{r=0}^{n} \tilde{B}_r W^{lr}, l = 0,1,\ldots,(2p-1)q$$

Step 4. Evaluate the generalized inverse

$$A(s)^{+} = \frac{B(s)}{-a_k(s)}$$

Drazin Inverse

For every matrix $A \in R^{m \times m}$, a unique matrix $A^D \in R^{m \times m}$, which is called Drazin inverse, exists satisfying

(i) $A^{k+1} A^D = A^k$ for $k = ind(A) = \min(k \in N : rank\left(A^k\right) = rank\left(A^{k+1}\right))$

(ii) $A^D A A^D = A^D$

(iii) $A A^D = A^D A$

Drazin Inverse

[Staminirovic and Karampetakis 2000] Consider a nonregular one-variable rational matrix $A(s)$. Assume that

$a(z,s) = \det\left[zI_m - A(s)\right] = a_0(s)z^m + a_1(s)z^{m-1} + \ldots + a_{m-1}(s)z + a_m(s)$ where

$a_0(s) \equiv 1, \quad z \in C$ is the characteristic polynomial of A(s) consider the following sequence of $m \times m$ polynomial matrices

$B_j(s) = a_0(s)A(s)^j + \ldots + a_{j-1}(s)A(s) + a_j(s)I_m, a_0(s) = 1, \quad j = 0, \ldots, m$

Let $a_m(s) \equiv 0, \ldots, a_{t+1}(s) \equiv 0, \quad a_t(s) \neq 0.$ Define the following

set $\Lambda = \{s_i \in C : a_t(s_i) = 0\}$ Also assume $B_m(s) \equiv 0, \ldots, B_r(s) = 0, B_{r-1}(s) \neq 0$ *and* k=r-t.

In the case that $s \in C \setminus \Lambda$ and $k > 0$, the Drazin inverse of $A(s)$ is given by

$$A^D = (-1)^{k+1} a_t(s)^{-k-1} A(s)^k B_{t-1}(s)^{k+1}$$

$$B_{t-1}(s) = a_0(s)A(s)^{t-1} + \ldots + a_{t-2}(s)A(s) + a_{t-1}(s)I_m$$

In the case $s \in C \setminus \Lambda$ and $k = 0$, we get $A(s)^D = O$.

Computation of the Drazin Inverse via DFT

Step 1 (Evaluation of a(s,z))

$$a(s,z)=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} a_{l_1,l_2} s^{l_1} z^{l_2} = \det\left[zI_m - A(s)\right]$$

- We use the following $R=(2mq+1)\times(m+1)$ points

  $u_i(r_i)=W_i^{-r_i}, i=1,2,\ W_i=e^{\frac{2\pi j}{M_i+1}} i=1,2; M_1=2mq; M_2=m$

- $\tilde{a}_{r_1,r_2}=\det[u_2(r_2)I_m - A(u_1(r_1))]=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} a_{l_1,l_2} W_1^{-r_1 l_1} W_2^{-r_2 l_2}$

  $[\tilde{a}_{r_1,r_2}]$ and $[a_{l_1,l_2}]$ form a DFT pair

- $a_{l_1,l_2}=\frac{1}{R}\sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2} \tilde{a}_{r_1,r_2} W_1^{r_1 l_1} W_2^{r_2 l_2} \quad ,l_1=0,1,\ldots,2mq,\quad l_2=0,1,..,m$

Step 2

Find $t: a_{t+1}(s) = a_{t+2}(s) = \ldots = a_m(s) = 0$

$a_t(s) \neq 0$ or $a_{r_1,0} = a_{r_1,1} == a_{r_1,t+1} = 0 \;\; \forall r_1$ and $a_{r_1,t} \neq 0$ for some t.

Step 3 ($Evaluate \ r \geq t : B_m(s) \equiv 0, \ldots, B_r(s) \equiv 0, B_{r-1}(s) \neq 0$
$B_j(s) = A(s)^j + a_1(s)A(s)^{j-1} + a_{j-1}(s)A(s) + a_j(s)I_m$)

$j = m$

Determine the value of $B_j(s)$ at the following $n_j + 1$ points (or any other $n_j + 1$ distinct points)

$$u(r) = W^{-r}, W = e^{\frac{2\pi j}{n_j+1}}$$

Do WHILE $(B_j(s) = 0 \ \ \forall u(r))$

$j = j - 1$

Determine the value of $B_j(s)$ at the following $n_j + 1$ points

$$u(r) = W^{-r}, W = e^{\frac{2\pi j}{n_j+1}}$$

END DO

$r = j$

Step 4 *(Evaluation of $A(s)^k B_{t-1}(s)^{k+1}$)*

- $B(s) = A(s)^k B_{t-1}(s)^{k+1} = \sum_{l=0}^{n} B_l s^l$,

  $B_{t-1}(s) = a_0(s)A(s)^{t-1} + \ldots + a_{t-2}(s)A(s) + a_{t-1}(s)I_m$

- We use the following (n+1) interpolation points

  $u(r) = W^{-r}, W = e^{\frac{2\pi j}{n+1}}$

- $\tilde{B}_r = \sum_{l=0}^{n} B_l W^{-lr}$

  $[\tilde{B}_i]$ and $[B_l]$ form a DFT pair

- $B_l = \frac{1}{R} \sum_{r=0}^{n} \tilde{B}_l W^{lr}$

Step 5 *(Evaluation of $a_t(s)^{k+1}$)*

- $$a(s) = a_t(s)^{k+1} = \sum_{l=0}^{n} a_l s^l$$

- We use the following (n+1) interpolation points

  $$u(r) = W^{-r}, W = e^{\frac{2\pi j}{n+1}}$$

- $$\tilde{a}_r = a(u(r)) = \sum_{l=0}^{n} a_l W^{-lr}$$

  $[\tilde{a}_l]$ and $[a_l]$ form a DFT pair

- $$a_l = \frac{1}{R} \sum_{r=0}^{n} \tilde{a}_r W^{lr}, \quad l = 0, 1, \ldots, n$$

Step 6. (Evaluation of the Drazin inverse)

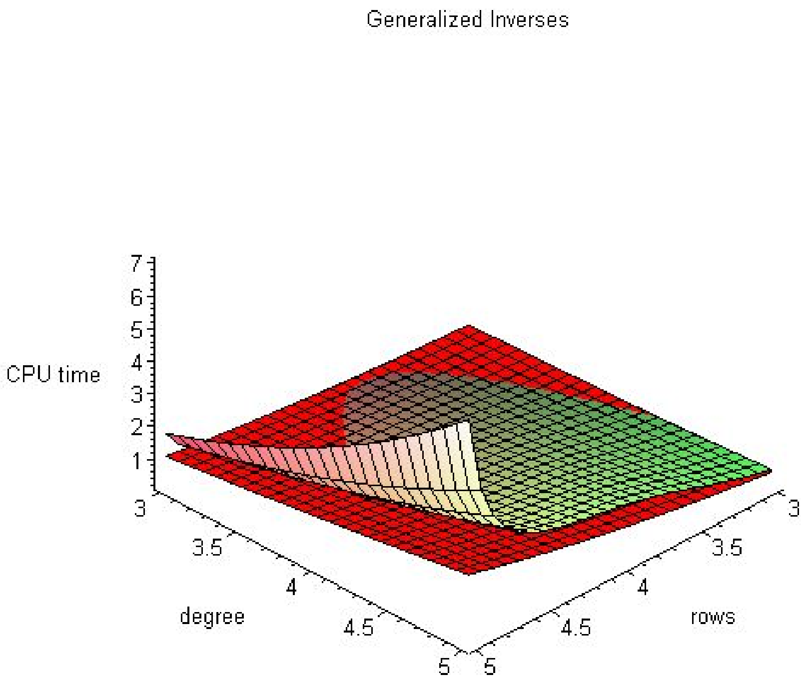
$$A(s)^D = \frac{B(s)}{a(s)}$$

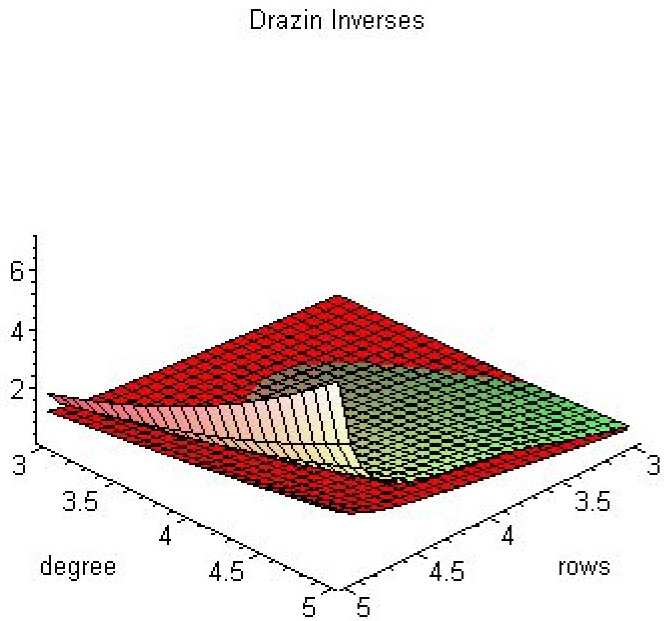
Implementation

- The above algorithms have been implemented in Mathematica.

- The following graphs shows the efficiency of the DFT based algorithms compared to the algorithms described in [Karampetakis 1997, Staminirovic and Karampetakis 2000]. The red surface represents the DFT based algorithms.

Graphs


Generalized Inverses
CPU time
7
6
5
4
3
2
1
3
3.5
4
degree
4.5
5
5
4.5
4
3.5
3
rows


Drazin Inverses
6
4
2
3
3.5
4
degree
4.5
5
5
4.5
4
3.5
3
rows

Conclusions

- Two new algorithms have been presented for the computation of the generalized inverse and Drazin inverse of a polynomial matrix.
- The proposed algorithms proved to be more efficient from the known ones in the case where the degree and the size of the polynomial matrix get bigger.
- The proposed algorithms can be easily extended to the multivariable polynomial matrices.