Linearizations of Polynomial Matrices with Symmetries and their Applications

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Outline

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- A new family of companion forms
- Applications to systems described by polynomial matrices with symmetries
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Preliminaries

Consider the polynomial matrix:

$$T(s) = T_n s^n + T_{n-1} s^{n-1} + \ldots + T_0,$$

where $T_i \in \mathbb{C}^{p \times p}, \det T(s) \neq 0, \text{for almost every } s \in \mathbb{C}$

The following matrix pencil is known as the ***1st companion matrix*** of $T(s)$

$$P(s) = s \begin{bmatrix} T_n & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_p \end{bmatrix} - \begin{bmatrix} -T_{n-1} & -T_{n-2} & \cdots & -T_0 \\ I_p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_p & 0 \end{bmatrix}$$

Preliminaries

The ***2nd companion matrix*** of $T(s)$ is accordingly defined as:

$$\hat{P}(s) = s \begin{bmatrix} T_n & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_p \end{bmatrix} - \begin{bmatrix} -T_{n-1} & I_p & \cdots & 0 \\ -T_{n-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & I_p \\ -T_0 & 0 & \cdots & 0 \end{bmatrix}$$

It can be easily seen that:

$$\det T(s) = \det P(s) = \det \hat{P}(s)$$

The two companion forms are **linearizations** of $T(s)$.

A new family of Companion Forms

In [ELA04] we have introduced a new family of linearizations. Define the matrices:

$$A_n = diag\{T_n, I_{p(n-1)}\},$$

$$A_k = \begin{bmatrix} I_{p(n-k-1)} & 0 & \cdots \\ 0 & C_k & \ddots \\ \vdots & \ddots & I_{p(k-1)} \end{bmatrix}, \quad k = 1, 2, \ldots, n-1,$$

where $C_k = \begin{bmatrix} -T_k & I_p \\ I_p & 0 \end{bmatrix}.$

$$A_0 = diag\{I_{p(n-1)}, -T_0\},$$

A new family of Companion Forms

Lemma [ELA04]. *The first and second companion forms of T(s) are given respectively by*

$$P(s) = sA_n - A_{n-1}A_{n-2}\ldots A_0,$$

$$\hat{P}(s) = sA_n - A_0\ldots A_{n-2}A_{n-1}.$$

Theorem [ELA04]. *Let* $P(s)$ *be the first companion form of a regular polynomial matrix* $T(s)$*. Then for every possible permutation* $(i_1, i_2, \ldots, i_n)$ *of the n-tuple* $(0, 1, \ldots, n-1)$ *the matrix pencil* $P(s)$ *is strictly equivalent to*

$$Q(s) = sA_n - A_{i_1}A_{i_2}\ldots A_{i_n}$$

A new family of Companion Forms

Example. *Let* $T(s) = T_3 s^3 + T_2 s^2 + T_1 s + T_0.$

We may choose:

$$A_3 = \begin{bmatrix} T_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad A_2 = \begin{bmatrix} -T_2 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$R(s) = sA_3 - A_2 A_1 A_0$$

$$A_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & -T_1 & I \\ 0 & I & 0 \end{bmatrix} \quad A_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -T_0 \end{bmatrix}$$

or

$$R(s) = s\begin{bmatrix} T_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} -T_2 & -T_1 & -T_0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

A new family of Companion Forms

Corollary [ELA04]. *Let $P(s)$ be the first companion form of a regular polynomial matrix $T(s)$. For any four ordered sets of indices*

$$I_k = (i_{k,1}, i_{k,2}, \ldots, i_{k,n_k}), k = 1,2,3,4$$

such that $I_i \cap I_j = \varnothing$ *for* $i \neq j$ *and* $\bigcup_{k=1}^{4} I_k = \{1,2,3,\ldots,n-1\}$ *the matrix pencil*

$$R(s) = sA_{I_1}^{-1} A_n A_{I_2}^{-1} - A_{I_3} A_0 A_{I_4},$$

is strictly equivalent to $P(s)$, *where* $A_{I_k} = A_{i_{k,1}} A_{i_{k,2}} \ldots A_{i_{k,n_k}}$ *for* $I_k \neq \varnothing$ *and* $A_{I_k} = I$ *for* $I_k = \varnothing$.

A new family of Companion Forms

Example. *Let* $T(s) = T_3 s^3 + T_2 s^2 + T_1 s + T_0.$

We may choose:

$$A_3 = \begin{bmatrix} T_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad A_2 = \begin{bmatrix} -T_2 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \Rightarrow A_2^{-1} = \begin{bmatrix} 0 & I & 0 \\ I & T_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$R(s) = sA_3A_2^{-1} - A_1A_0,$$

$$A_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & -T_1 & I \\ 0 & I & 0 \end{bmatrix} \quad A_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -T_0 \end{bmatrix}$$

or

$$R(s) = s\begin{bmatrix} 0 & T_3 & 0 \\ I & T_2 & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & -T_1 & -T_0 \\ 0 & I & 0 \end{bmatrix}.$$

Applications to systems described by polynomial matrices with symmetries

Definition. Let det(T_n)≠0. Define the following member of the new family of linearizations as follows:

$$R_s(s) = sA_{odd}^{-1} - A_{even},$$

where

$$A_{even} = A_0 A_2 \ldots A_n^{-1}, A_{odd} = A_1 A_3 \ldots A_{n-1}, \text{ for } n \text{ even}$$

and

$$A_{even} = A_0 A_2 \ldots A_{n-1}, A_{odd} = A_1 A_3 \ldots A_n^{-1}, \text{ for } n \text{ odd},$$

The constraint det(T_n)≠0 is needed in case where n is even. However, in that case we can get T_{n+1}=0 and we directly go the case where n is odd.

Applications to systems described by polynomial matrices with symmetries

Example. *We illustrate the form of $R_s(s)$ for n=4 respectively.*

$$T(s) = T_4 s^4 + T_3 s^3 + T_2 s^2 + T_1 s + T_0$$

$$R_s(s) = sA_{odd}^{-1} - A_{even} = s(A_1 A_3)^{-1} - A_0 A_2 A_4^{-1}$$

$$A_3 = \left[\begin{array}{cc|cc} -T_3 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ \hline 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{array}\right] \Rightarrow A_3^{-1} = \left[\begin{array}{cc|cc} 0 & I_p & 0 & 0 \\ I_p & T_3 & 0 & 0 \\ \hline 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{array}\right] \qquad A_1^{-1} = \left[\begin{array}{cc|cc} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ \hline 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & T_1 \end{array}\right]$$

$$A_0 = \left[\begin{array}{ccc|c} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ \hline 0 & 0 & 0 & -T_0 \end{array}\right] \qquad A_2 = \left[\begin{array}{c|cc|c} I_p & 0 & 0 & 0 \\ \hline 0 & -T_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ \hline 0 & 0 & 0 & I_p \end{array}\right] \qquad A_4 = \left[\begin{array}{c|ccc} T_4 & 0 & 0 & 0 \\ \hline 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{array}\right]$$

Applications to systems described by polynomial matrices with symmetries

Example. *We illustrate the form of $R_s(s)$ for $n=4$ respectively.*

$$R_s(s) = sA_{odd}^{-1} - A_{even} = s(A_1A_3)^{-1} - A_0A_2A_4^{-1}$$

$$R_s(s) = s\begin{bmatrix} 0 & I_p & & \\ I_p & T_3 & & \\ & & 0 & I_p \\ & & I_p & T_1 \end{bmatrix} - \begin{bmatrix} T_4^{-1} & & & \\ & -T_2 & I_p & \\ & I_p & 0 & \\ & & & -T_0 \end{bmatrix}$$

Applications to systems described by polynomial matrices with symmetries

Example. *We illustrate the form of $R_s(s)$ for n=5 respectively.*

$$R_s(s) = s\left(A_1 A_3 A_5^{-1}\right)^{-1} - \left(A_0 A_2 A_4\right)$$

$$R_s(s) = s\begin{bmatrix} T_5 & & & & \\ & 0 & I_p & & \\ & I_p & T_3 & & \\ & & & 0 & I_p \\ & & & I_p & T_1 \end{bmatrix} - \begin{bmatrix} -T_4 & I_p & & & \\ I_p & 0 & & & \\ & & -T_2 & I_p & \\ & & I_p & 0 & \\ & & & & -T_0 \end{bmatrix}$$

$$T(s) = 0s^5 + T_4 s^4 + T_3 s^3 + T_2 s^2 + T_1 s + T_0$$

Systems with Symmetric Coefficients

We consider systems described by differential equations of the form:

$$\sum_{i=0}^{n} T_i \frac{d^i x}{dt^i} = Bu$$

with symmetric coefficients, i.e. $T_i^\top = T_i$. Define:

$$T(s) = T_n s^n + T_{n-1} s^{n-1} + \ldots + T_0,$$

the polynomial matrix associated to the above system.

Question. Is there a linearization of $T(s)$ that preserves its symmetric structure?

Answer. The proposed linearization $R_s(s)$ has this appealing property.

Systems with Symmetric Coefficients

Example (2nd order system). *Consider the second order mechanical system described by*:

$$M\ddot{x} + C\dot{x} + Kx = Bu$$

Where the matrices M,C and K are symmetric. The associated polynomial matrix is:

$$T(s) = s^2 M + sC + K$$

The proposed symmetric linearization of $T(s)$ is:

$$R_s(s) = s\underbrace{\begin{bmatrix} 0 & I \\ I & C \end{bmatrix}}_{A_1} - \underbrace{\begin{bmatrix} I & 0 \\ 0 & -K \end{bmatrix}}_{A_0}\underbrace{\begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix}}_{A_2^{-1}} = \begin{bmatrix} -M^{-1} & sI \\ sI & sC + K \end{bmatrix}$$

Obviously, the coefficient matrices of the above pencil are symmetric too.

Systems with Symmetric Coefficients

Example (2nd order system). *Consider the second order mechanical system described by*:

$$M\ddot{x} + C\dot{x} + Kx = Bu$$

Where the matrices M,C and K are symmetric. The associated polynomial matrix is:

$$T(s) = 0s^3 + s^2M + sC + K$$

The proposed symmetric linearization of $T(s)$ is:

$$R_s(s) = s\left(A_1A_3^{-1}\right)^{-1} - \left(A_0A_2\right) = s\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}_{A_3}\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & C \end{bmatrix}}_{A_1^{-1}} - \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -K \end{bmatrix}}_{A_0}\underbrace{\begin{bmatrix} -M & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}_{A_2} = \begin{bmatrix} M & -I & 0 \\ -I & 0 & sI \\ 0 & sI & sC+K \end{bmatrix}$$

Obviously, the coefficient matrices of the above pencil are symmetric too.

Systems with Symmetric Coefficients

Example (3rd order system). *The numerical solution of vibration problems by the dynamic element method requires the solution of the cubic eigenvalue problem of the form* :

$$(\lambda^3 F_3 + \lambda^2 F_2 + \lambda F_1 + F_0)v = 0,$$

where $F_i = F_i^{\mathsf{T}},\ \ i = 0, 1, 2, 3.$ The proposed linearization is:

$$R_s(s) = \lambda \begin{bmatrix} F_3 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & F_1 \end{bmatrix} - \begin{bmatrix} -F_2 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & -F_0 \end{bmatrix}$$

Systems with Alternating Coefficients

Let $T(s) = T_n s^n + T_{n-1} s^{n-1} + \ldots + T_0,$

where the coefficients T_i alternate between symmetric and skew symmetric.

Definition:

- If $T_i^{\mathsf{T}} = (-1)^i T_i, \text{ for } i = 0,1,2,\ldots,n$

$$L(s) = M_2 R_s(s) M_3, \text{for } n \text{ even}$$

$$L(s) = M_3 R_s(s) M_4, \text{for } n \text{ odd}$$

- If $T_i^{\mathsf{T}} = (-1)^{i+1} T_i, \text{ for } i = 0,1,2,\ldots,n$

$$L(s) = M_3 R_s(s) M_4, \text{for } n \text{ even}$$

$$L(s) = M_2 R_s(s) M_3, \text{for } n \ odd$$

where $P_i = diag\left\{I_{(i-1)p}, -I_p, I_{(n-i)p}\right\}, 0 < i \le n, M_i = \prod_{j=0}^{\left\lfloor \frac{n-i}{4} \right\rfloor} P_{4j+i}$

Systems with Alternating Coefficients

Lemma. The matrix pencil $L(s)$ defined above is a linearization of the polynomial matrix $T(s)$, having the same alternating property with $T(s)$.

Example (Hamiltonian eigenvalue problems). *Consider the mechanical system governed by the differential equation:*

$$M\ddot{x} + C\dot{x} + Kx = Bu$$

The computation of the optimal control u, *that minimizes the cost functional:*

$$\int_{t_0}^{t_1} (x^{\top} Q_0 x + \dot{x}^{\top} Q_1 \dot{x} + u^{\top} R u) dt$$

Systems with Alternating Coefficients

is associated with the eigenvalue problem:

$$\left(\lambda^2\begin{bmatrix} M & 0 \\ -Q_1 & -M^\intercal \end{bmatrix} + \lambda\begin{bmatrix} C & 0 \\ 0 & C^\intercal \end{bmatrix} + \begin{bmatrix} K & -BR^{-1}B^\intercal \\ Q_0 & -K^\intercal \end{bmatrix}\right)\begin{bmatrix} v \\ w \end{bmatrix} = 0$$

The coefficient matrices are from left to right *Hamiltonian, skew Hamiltonian* and again *Hamiltonian*. After some manipulations the problem is transformed to the equivalent:

$$\left(\lambda^2\begin{bmatrix} Q_1 & M^\intercal \\ M & 0 \end{bmatrix} + \lambda\begin{bmatrix} 0 & -C^\intercal \\ C & 0 \end{bmatrix} + \begin{bmatrix} -Q_0 & K^\intercal \\ K & -BR^{-1}B^\intercal \end{bmatrix}\right)\begin{bmatrix} v \\ w \end{bmatrix} = 0$$

where now the coefficient matrices are respectively *symmetric, skew symmetric* and again *symmetric*

Systems with Alternating Coefficients

The proposed linearization of the above eigenvalue problem now takes the form:

$$L(s) = s\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & C^\intercal \\ 0 & -I & -C & 0 \end{bmatrix} - \begin{bmatrix} 0 & M^{-1} & 0 & 0 \\ M^{-\intercal} & -M^{-\intercal}Q_1M^{-1} & 0 & 0 \\ 0 & 0 & -Q_0 & K^\intercal \\ 0 & 0 & K & -BR^{-1}B^\intercal \end{bmatrix}$$

Systems with Alternating Coefficients

or, by taking a third order systems with zero highest coefficient matrix, the form:

$$L(s) = \underbrace{\begin{bmatrix} Q_1 & M^T & -I & 0 & 0 & 0 \\ M & 0 & 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 & sI & 0 \\ 0 & -I & 0 & 0 & 0 & sI \\ 0 & 0 & -sI & 0 & Q_0 & -K^T \\ 0 & 0 & 0 & -sI & -K & BR^{-1}B^T \end{bmatrix}}_{M_3R_s(s)}$$

Conclusions

- The new family of linearizations has important applications in
  - Systems with symmetric coefficients
  - Systems with alternating Symmetric – skew Symmetric coefficients
  - Systems with alternating Hamiltonian – skew Hamiltonian structure
- The proposed linearizations preserve the structure even in the general case (degree>2)
- The computational advantages of the proposed linearizations are subject of future research

2-D Systems with Symmetric Coefficients

Example *Consider the 2-D system described by*:

$$M\frac{\partial^2 x}{\partial t_1^2} + N\frac{\partial^2 x}{\partial t_2^2} = 0$$

Where the matrices M and N are symmetric. The associated polynomial matrix is:

$$T(z_1, z_2) = z_1^3 0 + z_1^2 M + z_1 0 + N z_2^2$$

The proposed symmetric linearization of $T(z_1,z_2)$ is:

$$R_s^1(z_1,z_2) = z_1\left(A_1 A_3^{-1}\right)^{-1} - \left(A_0 A_2\right) = z_1 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}_{A_3} \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}}_{A_1^{-1}} - \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -Kz_2^2 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} -M & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}_{A_2} = \begin{bmatrix} M & -I & 0 \\ -I & 0 & z_1 I \\ 0 & z_1 I & Kz_2^2 \end{bmatrix}$$

Obviously, the coefficient matrices of the above pencil are symmetric too.

2-D Systems with Symmetric Coefficients

$$R_s^1(z_1,z_2)=\begin{bmatrix} M & -I & 0 \\ -I & 0 & z_1 I \\ 0 & z_1 I & K z_2^2 \end{bmatrix} = \underbrace{\begin{bmatrix} M & -I & 0 \\ -I & 0 & z_1 I \\ 0 & z_1 I & 0 \end{bmatrix}}_{K'} + z_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C'} + z_2^2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{bmatrix}}_{M'} + z_2^3 0$$

The proposed symmetric linearization of the above polynomial matrix in z_2 is:

$$R_s^2(z_1,z_2)=\begin{bmatrix} M' & -I & 0 \\ -I & 0 & z_2 I \\ 0 & z_2 I & z_2 C' + K' \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 & 0 & -I & 0 & 0 & 0 \\ \hline -I & 0 & 0 & 0 & 0 & 0 & z_2 I & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & z_2 I & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & z_2 I \\ \hline 0 & 0 & 0 & z_2 I & 0 & 0 & M & -I & 0 \\ 0 & 0 & 0 & 0 & z_2 I & 0 & -I & 0 & z_1 I \\ 0 & 0 & 0 & 0 & 0 & z_2 I & 0 & z_1 I & 0 \end{array}\right]$$

Obviously, the coefficient matrices of the above pencil are symmetric too.

Realizations of 2nd order

Example (3rd order system). *The numerical solution of vibration problems by the dynamic element method requires the solution of the cubic eigenvalue problem of the form* :

$$(\lambda^3 F_3 + \lambda^2 F_2 + \lambda F_1 + F_0)v = 0,$$

where $F_i = F_i^{\mathsf{T}}, \; i = 0,1,2,3.$ The proposed realization of 2nd order is:

$$P(\lambda) = \lambda^3 0 + \lambda^2 \underbrace{(\lambda F_3)}_{M} + \lambda \underbrace{(\lambda F_2 + F_1)}_{C} + \underbrace{F_0}_{K}$$

$$R_s(\lambda) = \begin{bmatrix} M & -I & 0 \\ -I & 0 & \lambda I \\ 0 & \lambda I & \lambda C + K \end{bmatrix} = \begin{bmatrix} \lambda F_3 & -I & 0 \\ -I & 0 & \lambda I \\ 0 & \lambda I & \lambda(\lambda F_2 + F_1) + F_0 \end{bmatrix} = \begin{bmatrix} \lambda F_3 & -I & 0 \\ -I & 0 & \lambda I \\ 0 & \lambda I & \lambda^2 F_2 + \lambda F_1 + F_0 \end{bmatrix}$$