Inverses of Multivariable Polynomial Matrices by Discrete Fourier Transforms

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Outline of the presentation

- nD polynomial interpolation
  nD DFT
  Polynomial interpolation and DFT
- Computation of the Generalized inverse via DFT
- Computation of the Drazin inverse via DFT
- Complexity
  Perturbation analysis
- Conclusions

Objectives

- A **DFT** based algorithm for the evaluation of the **generalized inverse** and the **Drazin inverse** of a multivariable polynomial matrix is presented.
- The **efficiency** of the algorithms is illustrated via complexity analysis

Notation

Consider the polynomial matrix with real coefficients in the n indeterminates $z_1, z_2, \ldots, z_n$ (called nD polynomial matrix)

$$A(z_1, \ldots, z_n) = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} \cdots \sum_{k_n=0}^{M_n} \left(A_{k_1 \ldots k_n}\right) \times \left(z_1^{k_1} \ldots z_n^{k_n}\right) \in R\left[z_1, \ldots, z_n\right]^{p \times m}$$

with $A_{k_1 \ldots k_n} \in R^{p \times m}$. For brevity and when the variables are obvious, we will use the notation $\bar{z}$ instead of $\left(z_1, z_2, \ldots, z_n\right)$.

nD polynomial interpolation

Assume that we need to compute the value of *f(A)* where *f* some polynomial function. The evaluation-interpolation method of computing *f(A)* is based on the following steps

- **Step 1** Evaluation of the polynomial (matrix) at a set of $R = \prod_{i=1}^{n}(M_i + 1)$ suitably chosen points. This step results in a set of constant matrices.
- **Step 2** Application of the function f on the set of constant matrices derived from step 1.
- **Step 3** Computation of the coefficients of $f(A)$ through interpolation.

nD DFT

Consider the finite sequence $X(k_1,\dots,k_n)$ and $\tilde{X}(r_1,\dots,r_n)$, $k_i, r_i = 0,1,\dots,M_i$. In order for the sequence $X(k_1,\dots,k_n)$ and $\tilde{X}(r_1,\dots,r_n)$ to constitute a DFT pair the following relations should hold [Dudgeon] :

$$\tilde{X}(r_1,\dots,r_n) = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} \cdots \sum_{k_n=0}^{M_n} X(k_1,\dots,k_n) W_1^{-k_1 r_1} \cdots W_1^{-k_n r_n}$$

$$X(k_1,\dots,k_n) = \frac{1}{R} \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} \cdots \sum_{r_n=0}^{M_n} \tilde{X}(r_1,\dots,r_n) W_1^{k_1 r_1} \cdots W_1^{k_n r_n}$$

where

$$W_i = e^{\frac{2\pi j}{M_i+1}} \;\forall i = 1,2,3,\dots,n$$

$$R = \prod_{i=1}^{n} (M_i + 1)$$

and $X, \tilde{X}$ are discrete argument matrix-valued functions, with dimensions $p \times m$.

Polynomial interpolation and DFT

Efficiency of steps 1, 2 and 3 of the evaluation-interpolation algorithm.

• In step 2 a suitable algorithm computing *f(A)* where *A* is a constant matrix, must be chosen. Then, the critical points of the algorithm are steps 1 and 3.

• By using as evaluation points in **Step 1** *Fourier points*, the evaluation of the polynomial matrix is equivalent to the **DFT** of the multidimensional matrix of the coefficients. **Step 3** becomes an **inverse DFT** problem.

• By using FFT the speed of the algorithm improves dramatically.

Generalized Inverse of a Multivariable Polynomial Matrix

[Penrose] For every matrix $A \in R^{p \times m}$, a unique matrix $A^{+} \in R^{m \times p}$, which is called generalized inverse, exists satisfying

(i) $AA^{+}A = A$

(ii) $A^{+}AA^{+} = A^{+}$

(iii) $\left(AA^{+}\right)^{T} = AA^{+}$

(iv) $\left(A^{+}A\right)^{T} = A^{+}A$

where A^{T} denotes the transpose of A. In an analogous way we define the generalized inverse $A(z_1,\ldots,z_n)^{+} \in R\left(z_1,\ldots,z_n\right)^{m \times p}$ of a multivariable polynomial matrix

$A(z_1,\ldots,z_n) \in R\left[z_1,\ldots,z_n\right]^{p \times m}$

Generalized Inverse of a Multivariable Polynomial Matrix

Following the steps of [Karampetakis 1997] we have:

Let $A(\overline{z}) = A(z_1,\ldots,z_n) \in R\left[z_1,\ldots,z_n\right]^{p\times m}$ and

$$a(s,z_1,\ldots,z_n) = \det\left[sI_p - A(\overline{z})A(\overline{z})^T\right]$$
$$= \left(a_0(\overline{z})s^p + \cdots + a_{p-1}(\overline{z})s + a_p(\overline{z})\right)$$

$a_0(\overline{z}) = 1$, be the characteristic polynomial of $A(\overline{z}) \times A(\overline{z})^T$.

Let k such that it satisfies $a_p(\overline{z}) \equiv 0,\ldots,a_{k+1}(\overline{z}) \equiv 0$ while $a_k(\overline{z}) \neq 0,$ and define $\Lambda := \{(\overline{z}) \in C^n : a_k(\overline{z}) = 0\}.$

Then the generalized inverse $A(\overline{z})^+$ of $A(\overline{z})$ for $\overline{z} \in C^n - \Lambda$ is given by

$$A(\overline{z})^+ = -\frac{1}{a_k(\overline{z})} A(\overline{z})^T B_{k-1}(\overline{z})$$

$$B_{k-1}(\overline{z}) = a_0(\overline{z})\left[A(\overline{z})A(\overline{z})^T\right]^{k-1} + \cdots + a_{k-1}(\overline{z})I_p$$

Generalized Inverse of a Multivariable Polynomial Matrix Step 1

$$a(s,z_1,\ldots,z_n)=\sum_{k_0=0}^{b_0}\sum_{k_1=0}^{b_1}\ldots\sum_{k_n=0}^{b_n}\left(a_{k_0k_1..k_n}\right)\left(s^{k_0}z_1^{k_1}\ldots z_n^{k_n}\right)$$

- Define

$$\deg_s\left(a(s,z_1,\ldots,z_n)\right)=p:=b_0$$

$$\deg_{z_1}\left(a(s,z_1,\ldots,z_n)\right)\le 2pM_1:=b_1$$

$$\vdots$$

$$\deg_{z_n}\left(a(s,z_1,\ldots,z_n)\right)\le 2pM_n:=b_n$$

- We use following $R_1=\prod_{i=0}^{n}(b_i+1)$ interpolation points

$$u_i(r_j)=W_i^{-r_j};i=0,\ldots,n \text{ and } r_j=0,1,\ldots,b_i \text{ where } W_i=e^{\frac{2\pi j}{b_i+1}}$$

$$\tilde{a}_{r_0r_1..r_n}=\det\left[u_0(r_0)I_p-A(u_1(r_1),\ldots,u_n(r_n))\left[A(u_1(r_1),\ldots,u_n(r_n))\right]^T\right]=\sum_{l_0=0}^{b_0}\sum_{l_1=0}^{b_1}\ldots\sum_{l_n=0}^{b_n}\left(a_{l_0l_1..l_n}\right)\left(W_0^{-r_0l_0}\ldots W_n^{-r_nl_n}\right)$$

- $[a_{l_0l_1..l_n}]$ and $[\tilde{a}_{r_0r_1..r_n}]$ form a DFT pair   Thus we compute $a_{l_0l_1..l_n}$ via IDFT

Generalized Inverse of a Multivariable Polynomial Matrix Step 2

Find using a loop k such that $a_{k+1}(\overline{z}) = a_{k+2}(\overline{z}) = \cdots = a_p(\overline{z}) = 0$ and $a_k(\overline{z}) \neq 0$

Generalized Inverse of a Multivariable Polynomial Matrix Step 3

$$C(\overline{z}) = A(\overline{z})^T B_{k-1}(\overline{z}) = a_0(\overline{z})\left[A(\overline{z})A(\overline{z})^T\right]^{k-1} + \cdots + a_{k-1}(\overline{z}) I_p$$

- $C(\overline{z}) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_n=0}^{n_n} C_{l_0 \cdots l_n} \left(z_1^{l_1} \cdots z_n^{l_n} \right)$

  where $n_i = \max\left\{2(k-1)M_i + M_i, k = 1, \ldots, p\right\} = (2p-1)M_i$

- We use the following $R_2 = \prod_{i=1}^{n} \left\{(2p-1)M_i + 1\right\}$ points

  $$u_i(r_j) = W_i^{-r_j}; i = 1, \ldots, n \text{ and } r_j = 0, 1, \ldots, n_i \text{ where } W_i = e^{\frac{2\pi j}{n_i+1}}$$

  $$\tilde{C}_{r_1 \cdots r_n} = \sum_{l_1=0}^{n_1} \cdots \sum_{l_n=0}^{n_n} C_{l_0 \cdots l_n} W_1^{-r_1 l_1} \cdots W_n^{-r_n l_n}$$

- $\left[C_{l_0 \cdots l_n}\right]$ and $\left[\tilde{C}_{r_1 \cdots r_n}\right]$ form a DFT pair. Thus we compute $C_{l_0 \cdots l_n}$ via IDFT

Generalized Inverse of a Multivariable Polynomial Matrix Step 4

$$A(\overline{z})^{+} = -\frac{1}{a_k(\overline{z})}C(\overline{z})$$

Drazin inverse

For every matrix $A \in R^{m\times m}$, there exists a unique matrix $A^D \in R^{m\times m}$, which is called Drazin inverse, satisfying

(i) $A^D A^{k+1} = A^k$ for $k = ind(A) = \min(k \in N : rank\left(A^k\right) = rank\left(A^{k+1}\right))$

(ii) $A^D A A^D = A^D$

(iii) $A A^D = A^D A$

Drazin inverse

Consider a nonregular nD polynomial matrix $A(\overline{z})$. Assume that

$$a(s, z_1, \ldots, z_n) = \det\left[sI_m - A(\overline{z})\right] = \left(a_0(\overline{z})s^m + \cdots + a_{m-1}(\overline{z})s + a_m(\overline{z})\right)$$

where $a_0(\overline{z}) \equiv 1, z \in C$ is the characteristic polynomial of $A(\overline{z})$.

Also, consider the following sequence of $m \times m$ polynomial matrices

$$B_j(\overline{z}) = a_0(\overline{z})A(\overline{z})^j + \cdots a_{j-1}(\overline{z})A(\overline{z}) + a_j(\overline{z})I_m, a_0(\overline{z}) = 1, j = 0, \ldots, m$$

Let $a_m(\overline{z}) \equiv 0, \ldots, a_{t+1}(\overline{z}) \equiv 0, a_t(\overline{z}) \neq 0.$

Define the following set $\Lambda = \{\overline{z}_i \in C^n : a_t(\overline{z}_i) = 0\}$

Also, assume that $B_m(\overline{z}), \ldots, B_r(\overline{z}) = 0, B_{r-1}(\overline{z}) \neq 0$ and $k = r - t$.

In the case $\overline{z} \in C^n - \Lambda$ and $k > 0$, the Drazin inverse of $A(\overline{z})$ is given by

$$A(\overline{z})^D = \frac{A(\overline{z})^k B_{t-1}(\overline{z})^{k+1}}{a_t(\overline{z})^{k+1}}$$

$$B_{t-1}(\overline{z}) = a_0(\overline{z})A(\overline{z})^{t-1} + \cdots + a_{t-2}(\overline{z})A(s) + a_{t-1}(\overline{z})I_m$$

In the case $\overline{z} \in C^n - \Lambda$ and $k = 0$, we get $A(\overline{z})^D = O$.
For $\overline{z}_i \in \Lambda$ we can use the same algorithm again.

Drazin Inverse of a Multivariable Polynomial Matrix Step 1

$$a(s, z_1, \ldots, z_n) = \sum_{k_0=0}^{b_0} \sum_{k_1=0}^{b_1} \cdots \sum_{k_n=0}^{b_n} \left(a_{k_0 k_1 \ldots k_n}\right)\left(s^{k_0} z_1^{k_1} \ldots z_n^{k_n}\right)$$

- Define
$$\deg_s\left(a(s, z_1, \ldots, z_n)\right) = m := b_0$$
$$\deg_{z_1}\left(a(s, z_1, \ldots, z_n)\right) \leq mM_1 := b_1$$
$$\vdots$$
$$\deg_{z_n}\left(a(s, z_1, \ldots, z_n)\right) \leq mM_n := b_n$$

- We use following $R_1 = \prod_{i=0}^{n} (b_i + 1)$ interpolation points

$$u_i(r_j) = W_i^{-r_j}; i = 0, \ldots, n \text{ and } r_j = 0, 1, \ldots, b_i \text{ where } W_i = e^{\frac{2\pi j}{b_i + 1}}$$

$$\tilde{a}_{r_0 r_1 \ldots r_n} = \det\left[u_0(r_0) I_p - A(u_1(r_1), \ldots, u_n(r_n))\right]$$

- $[a_{l_0 l_1 \ldots l_n}]$ and $[\tilde{a}_{r_0 r_1 \ldots r_n}]$ form a DFT Thus we compute $a_{l_0 l_1 \ldots l_n}$ via IDFT

Drazin Inverse of a Multivariable Polynomial Matrix Step 2

Find using a loop t such that $a_{t+1}(\overline{z}) = a_{t+2}(\overline{z}) = \cdots = a_m(\overline{z}) = 0$ and $a_t(\overline{z}) \neq 0$

A polynomial matrix

$$B(z_1,\ldots,z_n) \in R[z_1,\ldots,z_n]^{m\times m}$$

of degrees q_i in respect with variables $z_i, i = 1,\ldots,n$, is the zero polynomial matrix iff its values at $R = \prod_{i=0}^{n}(q_i + 1)$ distinct points are the zero matrix.

Find r such that $B_m(\overline{z}),\ldots,B_r(\overline{z}) = 0, B_{r-1}(\overline{z}) \neq 0$ and $k = r - t$ using the above lemma.

Drazin Inverse of a Multivariable Polynomial Matrix Step 3

$$C(\overline{z}) = A(\overline{z})^k B_{t-1}(\overline{z})^{k+1}$$

- $$C(\overline{z}) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_n=0}^{n_n} C_{l_0 \cdots l_n} \left(z_1^{l_1} \cdots z_n^{l_n} \right), \; n_i = (t-1)(k+1)M_i$$
- We use the following $R_2 = \prod_{i=1}^{n} \{(n_i + 1)\}$ points

$$u_i(r_j) = W_i^{-r_j}; i = 1, \ldots, n \text{ and } r_j = 0, 1, \ldots, n_i \text{ where } W_i = e^{\frac{2\pi j}{n_i + 1}}$$

$$\tilde{C}_{r_1 \cdots r_n} = \sum_{l_1=0}^{n_1} \cdots \sum_{l_n=0}^{n_n} C_{l_0 \cdots l_n} W_1^{-r_1 l_1} \cdots W_n^{-r_n l_n}$$

- $\left[C_{l_0 \cdots l_n} \right]$ and $\left[\tilde{C}_{r_1 \cdots r_n} \right]$ form a DFT pair. Thus we compute $C_{l_0 \cdots l_n}$ via IDFT

Drazin Inverse of a Multivariable Polynomial Matrix Step 4

$$c(\overline{z}) = a_t(\overline{z})^{k+1}$$

- $$c(\overline{z}) = \sum_{k_1=0}^{d_1} \cdots \sum_{k_n=0}^{d_n} \left(c_{k_1..k_n}\right)\left(z_1^{k_1} \ldots z_n^{k_n}\right), \text{ where } d_i = tM_i(k+1)$$
- We use the following $R_3 = \prod_{i=1}^{n}(d_i + 1)$ points

$$u_i(r_j) = W_i^{-r_j}; i = 1, \ldots, n \text{ and } r_j = 0,1,\ldots,d_i, \text{ where } W_i = e^{\frac{2\pi j}{d_i+1}}$$

$$\tilde{c}_{r_1..r_n} = \sum_{l_1=0}^{d_1} \cdots \sum_{l_n=0}^{d_n} \left(c_{k_1..k_n}\right)\left(W_1^{-r_1 l_1} \ldots W_n^{-r_n l_n}\right)$$

- $\left[c_{l_0 \cdots l_n}\right]$ and $\left[\tilde{c}_{r_1 \cdots r_n}\right]$ form a DFT pair. Thus we compute $c_{l_0 \cdots l_n}$ via IDFT

Drazin Inverse of a Multivariable Polynomial Matrix
Step 5

$$A(\overline{z})^D = \frac{C(\overline{z})}{c(\overline{z})}$$

Complexity

- **Generalized inverse**

$O\left(mp^3 R_1 L_1\right)$ where

$$L_1 = (p+1) + \sum_{i=1}^{n} \log(2pM_i + 1) \text{ and } R_1 = (p+1)\prod_{i=1}^{n}(2pM_i + 1)$$

Complexity

- **Drazin inverse**

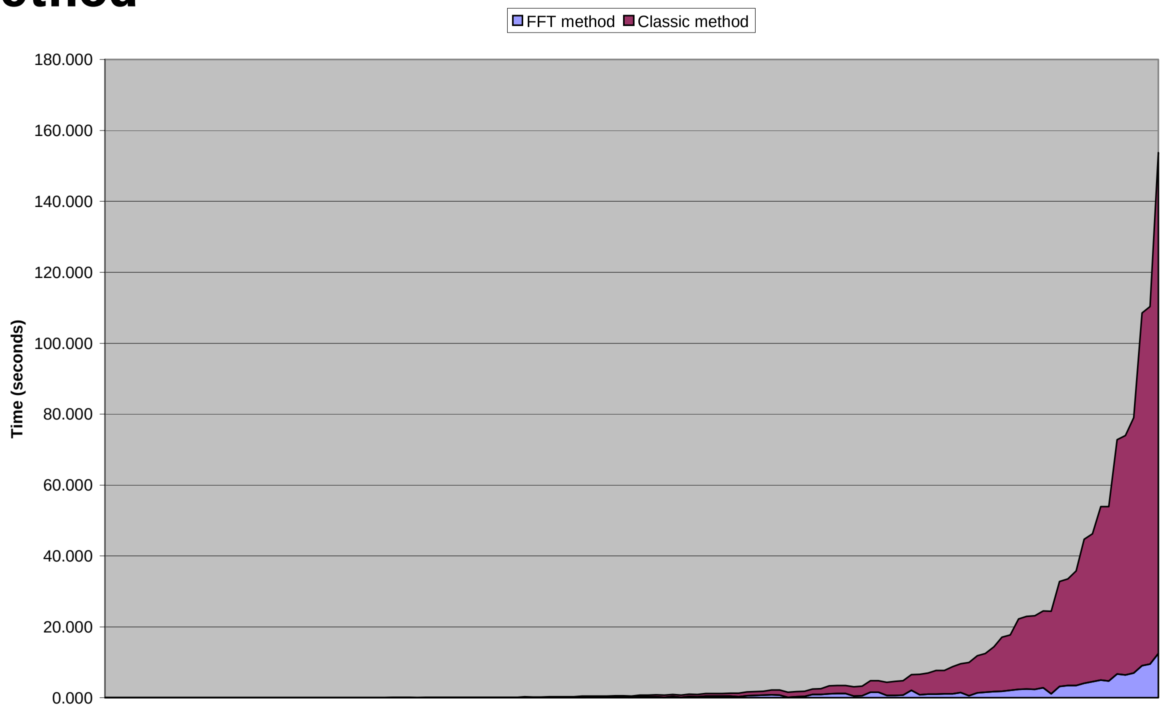
$$O\left(m^4 RL\right) \text{ where}$$

$$R = \max\{R_1, R_2, R_3\}, L = \max\{L_1, L_2, L_3\}$$

$$\text{and } L_1 = \sum_{i=0}^{n} \log(b_i + 1), L_2 = \sum_{i=1}^{n} \log(n_i + 1), L_3 = \sum_{i=1}^{n} \log(d_i + 1)$$

Numerical aspects of the algorithm

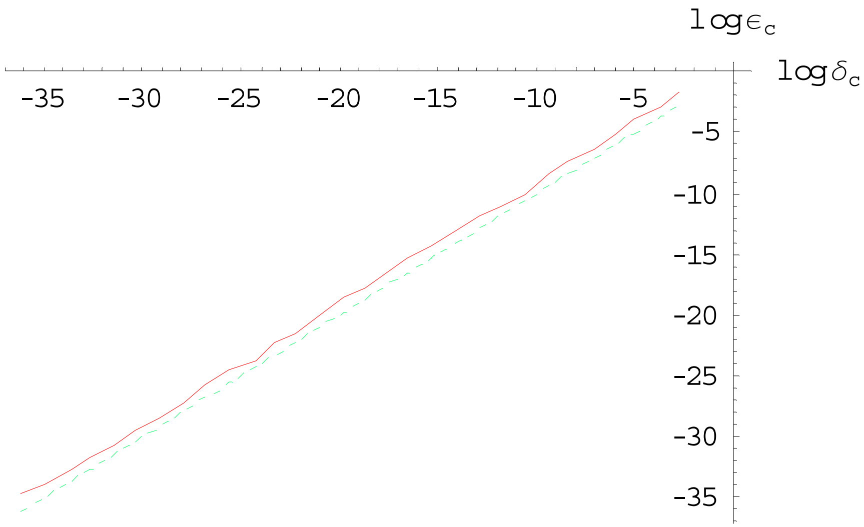
Comparison in Mathematica of the FFT and the symbolic method



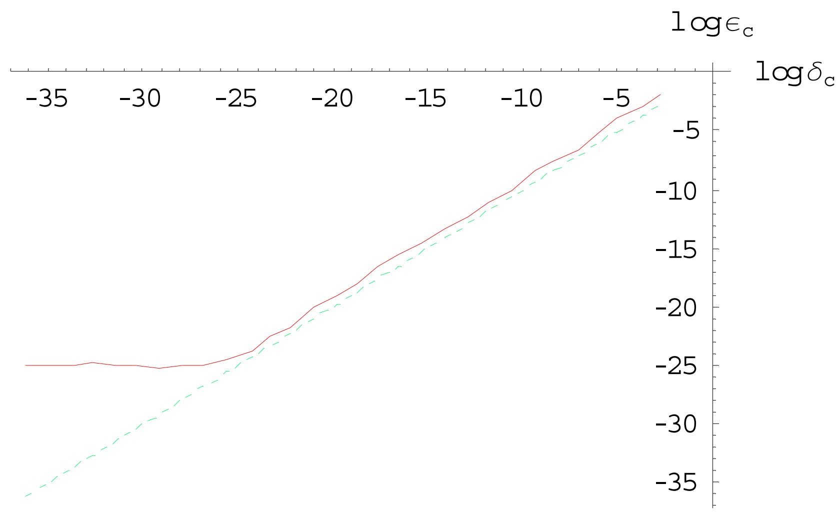
Random two variable polynomial matrices of dimensions up to **5x5** and degrees up to **8.**

Perturbation analysis

Relationship between the relative error $\varepsilon_c = \frac{\left\| \left(A(\bar{z})+\delta_A(\bar{z},c)\right)^+ - A^+(\bar{z}) \right\|_2}{\left\| A^+(\bar{z}) \right\|_2}$ and the relative error in the data $\delta_c = \frac{\left\| \delta_A(\bar{z},c) \right\|_2}{\left\| A(\bar{z}) \right\|_2}$, where $\left\| A(\bar{z}) \right\|_2$ denotes the 2-norm of the coefficient matrix $\begin{bmatrix} A_{0\ldots0} & \cdots & A_{M_1\ldots M_n} \end{bmatrix}$ of $A(\bar{z})$.



Perturbation analysis of the denominator polynomial of the generalized inverse



Perturbation analysis of the numerator of the generalized inverse

Conclusions

- In this paper two algorithms have been presented for determining the Moore-Penrose and Drazin inverse of nD polynomial matrices.
- The algorithms are based on the fast Fourier transform and therefore have the main advantages of speed in contrast to other known algorithms.
- Applications include model matching, the solution of multivariable Diophantine equations and its application to control system synthesis problems, etc.
- The above mentioned algorithms may be easily extended in order to determine other kind of inverses such as {2}, {1,2}, {1,2,3} and {1,2,4} inverses of multivariable polynomial matrices.