Notions of equivalence for discrete time AR representations

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Objectives

- We develop a new equivalence transformation between square polynomial matrices having as invariants both *finite* and *infinite* elementary divisors.
- This new transformation belongs to the same equivalence class with strict equivalence presented in (Vardulakis 2001).

Finite elementary divisors

Definition 1. Let $A(\sigma) \in R[\sigma]^{r \times r}$, $rank_{R(\sigma)} A(\sigma) = r$ have ℓ

distinct zeros: $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in C$

Let

$$S_{A(\sigma)}^{\lambda_i}(\sigma) = diag\left[(\sigma - \lambda_i)^{m_{i1}} (\sigma - \lambda_i)^{m_{i2}} \cdots (\sigma - \lambda_i)^{m_{ir}} \right]$$

be the **local** Smith form of $A(\sigma)$ at $\sigma = \lambda_i$

$$0 \le m_{i1} \le m_{i2} \le \ldots \le m_{ir}$$

$(\sigma - \lambda_i)^{m_{ij}}$, $i = 1, 2, \ldots, \ell$ $j = 1, 2, \ldots, r$ are the **finite elementary divisors** (FEDs) of $A(\sigma)$ at $\sigma = \lambda_i$

$m_{ij} \in Z^+$, are the **partial multiplicities** of λ_i

$m_i := \sum_{j=1}^{r} m_{ij}$, is the **multiplicity** of λ_i, $n := \sum_{j=1}^{\ell} m_i$

Infinite elementary divisors

If $A_0 \neq 0$ the **dual** $\tilde{A}(\sigma)$ of $A(\sigma)$ is defined:

$$\tilde{A}(\sigma) := \sigma^q A(\sigma^{-1}) = A_0\sigma^q + A_1\sigma^{q-1} + \ldots + A_q$$

$$rank\tilde{A}(0) = rank A_q < r \Rightarrow \tilde{A}(\sigma) \text{ has zeros at } \sigma = 0$$

Let

$$S^0_{\tilde{A}(\sigma)}(\sigma) = diag\left[\sigma^{\mu_1}, \sigma^{\mu_2}, \ldots, \sigma^{\mu_r}\right] (\text{local Smith form at } \sigma = 0)$$

$$0 \le \mu_1 \le \mu_2 \le \ldots \le \mu_r$$

Definition 2. The **infinite elementary divisors** (IEDs) of $A(\sigma)$ are the FEDs:

$$\sigma^{\mu_j}, \quad j = 1,2,\ldots,r$$

of $\tilde{A}(\sigma)$ at $\sigma = 0$. $\qquad \mu := \sum_{j=1}^{r} \mu_j = \#\text{ of IEDS}$

(Praagman 1991) $\qquad n + \mu = \mathrm{rq}$

Example (AutoRegressive representations)

$$A(s)=\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix}\qquad S^{\infty}_{A(s)}=\begin{bmatrix}1 & 0\\ 0 & s^3\end{bmatrix}\qquad S^{C}_{A(s)}=\begin{bmatrix}1 & 0\\ 0 & s+1\end{bmatrix}$$

Consider the q-th order discrete time AutoRegressive representation

$$A(\sigma)\xi_\kappa=0 \Leftrightarrow \mathrm{A}_0\xi_\kappa+\cdots+\mathrm{A}_q\xi_{\kappa+q}=0$$

$$B^N_{A(\sigma)}=\left\langle \begin{matrix}\underbrace{\begin{pmatrix}1\\ -1\end{pmatrix}(-1)^k,}_{due\ to\ S^C_{A(s)}(s)}\\ \\ \underbrace{\begin{pmatrix}\delta_{N-k}\\ 0\end{pmatrix},\begin{pmatrix}\delta_{N-k+1}\\ 0\end{pmatrix},\begin{pmatrix}\delta_{N-k+2}\\ -\delta_{N-k}\end{pmatrix}}_{due\ to\ S^0_{A(s)}(s)}\end{matrix}\right\rangle$$

Example (output zeroing problem)

Consider the polynomial matrix description

$$(\sigma^2)\xi_k = -u_k$$

$$y_k = (\sigma + 1)\xi_k$$

In order to find the state-input pair which gives rise to zero output (output zeroing problem) we have to solve the following system of difference equations

$$\underbrace{\begin{bmatrix} \sigma^2 & 1 \\ \sigma + 1 & 0 \end{bmatrix}}_{P(\sigma)} \underbrace{\begin{bmatrix} \xi_k \\ u_k \end{bmatrix}}_{x_k} = 0_{2\times 1}$$

$$\begin{pmatrix} \xi_k \\ u_k \end{pmatrix} = \begin{pmatrix} -l_1(-1)^k - l_4\delta_{N-k} \\ \\ l_1(-1)^k + l_2\delta_{N-k} + \\ +l_3\delta_{N-k+1} + l_4\delta_{N-k+2} \end{pmatrix}$$

Extended Unimodular Equivalence

Definition 3. (Pugh & Shelton 1978, Smith 1981)

Two polynomial matrices $A_1(s), A_2(s)$ are said to be *extended unimodular equivalent* (e.u.e.) if there exists polynomial matrices $M(s), N(s)$ such that :

$$M(s)A_1(s) = A_2(s)N(s) \text{ or } \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0$$

where the compound matrices $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$ are left and right prime respectively.

Theorem 1. (Pugh & Shelton 1978, Smith 1981)

•E.u.e. is an equivalence relation,

•Two square and nonsingular polynomial matrices are e.u.e. iff they possess the same finite elementary divisors.

Example

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{M(s)}\underbrace{\begin{bmatrix}1 & s^2\\0 & s+1\end{bmatrix}}_{A_1(s)}=\underbrace{\begin{bmatrix}1 & s^3\\0 & s+1\end{bmatrix}}_{A_2(s)}\underbrace{\begin{bmatrix}1 & s^2-s^3\\0 & 1\end{bmatrix}}_{N(s)}$$

E.u.e.

$$S^C_{A_1(s)}(s)=\begin{bmatrix}1 & 0\\0 & s+1\end{bmatrix}=S^C_{A_2(s)}(s)$$

Same f.e.d.

$$S^0_{\tilde{A}_1(s)}(s)=S^0_{\begin{bmatrix}s^2 & 1\\0 & s+s^2\end{bmatrix}}(s)=\begin{bmatrix}1 & 0\\0 & s^3\end{bmatrix}$$

Different i.e.d.

$$S^0_{\tilde{A}_2(s)}(s)=S^0_{\begin{bmatrix}s^3 & 1\\0 & s^2+s^3\end{bmatrix}}(s)=\begin{bmatrix}1 & 0\\0 & s^5\end{bmatrix}$$

Strict Equivalence

Definition 4.

Two polynomial matrices A_1(s),A_2(s) are said to be strict *equivalent* (s.e.) if there exist constant square nonsingular matrices M, N such that :

$$MA_1(s) = A_2(s)N \text{ or } \begin{bmatrix} M & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N \end{bmatrix} = 0$$

Note

Strict equivalence preserves both finite and infinite elementary divisors but it only relates matrices of the same dimensions.

A set of polynomial matrices

Praagman (1991), Gohberg et.al. (1982) $\quad n+\mu=\mathrm{rq}$

$$R_C[s]=\left\{A(s)\in R[s]^{r\times r}, r\in N-\{1\}, \det\left[A(s)\right]\neq 0, c=rq\right\}$$

$$A_1(s)=\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix};\; A_2(s)=\begin{bmatrix}1 & s^3\\ 0 & s+1\end{bmatrix};\; A_3(s)=\begin{bmatrix}s & 0 & -1 & 0\\ 0 & s & 0 & -1\\ 1 & 0 & 0 & s\\ 0 & 1 & 0 & 1\end{bmatrix}$$

$$r_1q_1=2\times 2=4 \qquad r_2q_2=2\times 3=6 \qquad r_3q_3=4\times 1=4$$

$$A_1(s), A_3(s)\in R_4[s] \qquad A_2(s)\in R_6[s]$$

A new equivalence transformation

Definition 5. Two polynomial matrices $A_1(s), A_2(s)$ are said to be *divisor equivalent* (d.e.) if there exists polynomial matrices $M(s), N(s)$ such that :

$$M(s)A_1(s) = A_2(s)N(s) \text{ or } \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0$$

where

(i) the compound matrices $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$ are left and right prime respectively,

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(ii) the compound matrices $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$ have no infinite elementary divisors.

Properties of divisor equivalence

Theorem 2. Two polynomial matrices $A_1(s), A_2(s) \in R_c[s]$ possess the same finite and infinite elementary divisors iff they are divisor equivalent.

Theorem 3.

(i) Divisor equivalence is an equivalence relation on $R_c[s]$.

(ii) Let $sE_1 - A_1, sE_2 - A_2 \in R[s]^{r\times r}$ with $\det\left[sE_i - A_i\right] \neq 0$. Then $sE_1 - A_1, sE_2 - A_2$ are strict equivalent i.e. $M(sE_1 - A_1)N = sE_2 - A_2$ with M, N nonsingular, iff they are divisor equivalent.

Remarks.

•The first condition (left and right primeness) ensures that the finite elementary divisors remain the same, while the second (absence of infinite elementary divisors) ensures the invariance of infinite elementary divisors.

Example

$$\underbrace{\begin{bmatrix} s+1 & 0 \\ s^2 & 0 \\ 0 & -1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} s^2 & 1 \\ 0 & s^3 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} s^2 & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s^2 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ s^3 & s+1 \\ 0 & -s \end{bmatrix}}_{N(s)}$$

(i) $S^C_{[M(s) \quad A_2(s)]}(s) = [I_3 \quad 0_{3,2}]$ $\qquad S^C_{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}}(s) = \begin{bmatrix} I_2 \\ 0_{3,2} \end{bmatrix}$

(ii) $M_1(s) = [M(s) \quad \tilde{A}_2(s)] = \begin{bmatrix} s+s^2 & 0 & 1 & s^2 & 0 \\ 1 & 0 & 0 & s & s^2 \\ 0 & -s^2 & 0 & 0 & 1 \end{bmatrix}$ with $S^0_{M_1(s)}(s) = [I_3 \quad 0_{3,2}]$

$$N_1(s) = \begin{bmatrix} \tilde{A}_1(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} s & s^3 \\ 0 & 1 \\ -s^3 & 0 \\ -1 & -s^2-s^3 \\ 0 & s^2 \end{bmatrix} \text{ with } S^0_{N_1(s)}(s) = \begin{bmatrix} I_2 \\ 0_{3,2} \end{bmatrix}$$

Example

$$A_1(s) = \begin{bmatrix} s^2 & 1 \\ 0 & s^3 \end{bmatrix} \; ; \; A_2(s) = \begin{bmatrix} s^2 & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s^2 \end{bmatrix}$$

Same f.e.d

$$S^C_{A_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{s^5} \end{bmatrix} \; ; \; S^C_{A_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & \boxed{s^5} \end{bmatrix}$$

Same i.e.d

$$S^0_{\tilde{A}_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{s} \end{bmatrix} \; ; \; S^0_{\tilde{A}_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & \boxed{s} \end{bmatrix}$$

Matrix pencil divisor equivalents of a polynomial matrix

Define

$$A(s) = A_0 + A_1 s + \cdots + A_q s^q \in R[s]^{r\times r}$$

and

$$sE - A := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \in R[s]^{rq\times rq}$$

Matrix pencil divisor equivalents of a polynomial matrix

Theorem 4. The polynomial matrix $A(s)$ and the pencil $sE - A$ are divisor equivalent.

$$\underbrace{\begin{bmatrix} 0_{(q-1)r,r} \\ sI_r - J \end{bmatrix}}_{M(s)} A(s) = \underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} \underbrace{\begin{bmatrix} (sI_r - J) \\ (sI_r - J)s \\ \vdots \\ (sI_r - J)s^{q-1} \end{bmatrix}}_{N(s)}$$

$$\underbrace{\begin{bmatrix} -(sI_r - J)s^{q-2}E_0(s) & -(sI_r - J)s^{q-3}E_1(s) & \cdots & -(sI_r - J)E_{q-2}(s) & -(sI_r - J)s^{q-1} \end{bmatrix}}_{M(s)} \times$$

$$\underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} = A(s) \times \underbrace{\begin{bmatrix} 0_{r,(q-1)r} & (sI_r - J) \end{bmatrix}}_{N(s)}$$

$sI_r - J = (s - s_0)I_r$ with $s_0 : \det[A(s_0)] \neq 0$

$$E_i(s) = E_{i-1}(s) + A_i s \qquad i = 0,1,\ldots,q-2$$
$$E_0 = A_0$$

Remarks

- A(s) and (sE-A) are d.e. and therefore have the same f.e.d. and i.e.d.
- The above reduction algorithm still remains the same for nonsquare polynomial matrices with the same property of preserving both the f.e.d. and i.e.d. .

Example

$$A(s)=\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix}=\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{A_0}+\underbrace{\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}}_{A_1}s+\underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{A_2}s^2$$

$$sE-A=\begin{bmatrix}sI_2 & -I_2\\ A_0 & sA_2+A_1\end{bmatrix}=\begin{bmatrix}s & 0 & -1 & 0\\ 0 & s & 0 & -1\\ 1 & 0 & 0 & s\\ 0 & 1 & 0 & 1\end{bmatrix}$$

Select s_0 such that $\det[A(s_0)]\neq 0$, let for example $s_0=2$ and define

$$sI_2-J=(s-2)I_2=\begin{bmatrix}s-2 & 0\\ 0 & s-2\end{bmatrix}$$

Matrix pencil divisor equivalents of a polynomial matrix

Example

$$\underbrace{\begin{bmatrix} 0_{2,2} \\ sI_2 - J \end{bmatrix}}_{M(s)} A(s) = (sE - A) \underbrace{\begin{bmatrix} (sI_2 - J) \\ (sI_2 - J)s \end{bmatrix}}_{N(s)}$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s-2 & 0 \\ 0 & s-2 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A(s)} = \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{sE-A} \underbrace{\begin{bmatrix} s-2 & 0 \\ 0 & s-2 \\ (s-2)s & 0 \\ 0 & (s-2)s \end{bmatrix}}_{N(s)}$$

$$\underbrace{\begin{bmatrix} -(sI_2 - J)A_0 & (sI_2 - J)s \end{bmatrix}}_{M(s)} (sE - A) = A(s) \underbrace{\begin{bmatrix} 0_{2,2} & (sI_2 - J) \end{bmatrix}}_{N(s)}$$

$$\underbrace{\begin{bmatrix} -(s-2) & 0 & (s-2)s & 0 \\ 0 & -(s-2) & 0 & (s-2)s \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{sE-A} = \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A(s)} \underbrace{\begin{bmatrix} 0 & 0 & s-2 & 0 \\ 0 & 0 & 0 & s-2 \end{bmatrix}}_{N(s)}$$

Matrix pencil divisor equivalents of a polynomial matrix

Example

$$A(s)=\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix} \;;\; sE-A=\begin{bmatrix}s & 0 & -1 & 0\\ 0 & s & 0 & -1\\ 1 & 0 & 0 & s\\ 0 & 1 & 0 & 1\end{bmatrix}$$

Same f.e.d.

$$S^{C}_{A(s)}(s)=\begin{bmatrix}1 & 0\\ 0 & s+1\end{bmatrix} \;;\; S^{C}_{(sE-A)}(s)=\begin{bmatrix}I_3 & 0\\ 0 & s+1\end{bmatrix}$$

$$\tilde{A}(s)=\begin{bmatrix}s^2 & 1\\ 0 & s+s^2\end{bmatrix} \;;\; E-sA=\underbrace{\begin{bmatrix}1 & 0 & -s & 0\\ 0 & 1 & 0 & -s\\ s & 0 & 0 & 1\\ 0 & s & 0 & s\end{bmatrix}}_{sE-A}$$

Same i.e.d.

$$S^{0}_{\tilde{A}(s)}(s)=\begin{bmatrix}1 & 0\\ 0 & s^3\end{bmatrix} \;;\; S^{0}_{(E-sA)}(s)=\begin{bmatrix}I_3 & 0\\ 0 & s^3\end{bmatrix}$$

Strict equivalence & divisor equivalence

Theorem 5. (Vardulakis & Antoniou,2001) Two polynomial matrices $A_1(s), A_2(s) \in R_c[s]$ are called strictly equivalent iff their equivalent matrix pencils $sE_1\text{-}A_1$ and $sE_2\text{-}A_2$ defined as previously are strict equivalent.

Theorem 6. Strict equivalence and divisor equivalence belong to the same equivalence class.

Geometrical interpretations of d.e.

Consider the discrete time AR - representation :

$$A(\sigma)\beta(k)=0 \quad \sigma\beta(k)=\beta(k+1) \quad A(\sigma)\in R[s]^{r\times r}$$

$$B=\{\beta(k): A(\sigma)\beta(k)=0, k\in[0,N]\}$$

then (Antoniou et. al.1998)

$$\dim B = n+\mu$$

$$M(\sigma)\mathrm{A}_1(\sigma)=\mathrm{A}_2(\sigma)\mathrm{N}(\sigma)\overset{\times\beta_1(\kappa)}{\Rightarrow}$$

$$M(\sigma)\mathrm{A}_1(\sigma)\beta_1(\kappa)=\mathrm{A}_2(\sigma)\mathrm{N}(\sigma)\beta_1(\kappa)$$

$$0=\mathrm{A}_2(\sigma)[\mathrm{N}(\sigma)\beta_1(\kappa)]\Rightarrow$$

$$\exists\beta_2(\kappa)=\mathrm{N}(\sigma)\beta_1(\kappa)$$

Conclusions

- A new equivalence between square polynomial matrices, named d.e. has been defined, with the property of preserving both the f.e.d. and i.e.d. of polynomial matrices (extension of strict equivalence).
- A known linearization algorithm has been viewed under the prism of the new transformation of d.e. .
- Further research still remains as concerns : the geometrical meaning of d.e., the extension of d.e. to system matrices, the study of d.e. to nonsquare polynomial matrices.
- D.e. belongs to the same equivalence class with strict equivalence presented in (Vardulakis 2001).