On the fundamental matrix of the inverse of a polynomial matrix and applications

N. P. Karampetakis
S. Vologiannidis
Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece
http://anadrasis.math.auth.gr

Contents

- The fundamental matrix of the inverse of a matrix pencil and applications
- Problem statement
- Computation of the fundamental matrix of the inverse of a polynomial matrix and its properties
- Applications to the solution of ARMA models
- Conclusions

Forward fundamental matrix of regular matrix pencils

Definition. [Mertzios, 1989] If $E,\ A \in \mathbb{C}^{n\times n}$ and $zE - A$ is invertible for some $z \in \mathbb{C}$, then for some $R_1 > 0$ and $|z| > R_1$, we have the Laurent expansion at infinity of

$$(zE - A)^{-1} = \Phi_{-\mu} z^{\mu-1} + \cdots + \Phi_0 z^{-1} + \Phi_1 z^{-2} + \cdots = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}$$

where the coefficient matrices $\Phi_i \in \mathbb{C}^{n\times n}$ are independent of z and are uniquely determined by E, A. $\Phi_i,\ i = -\mu, -\mu+1, ..$ are known as the *(forward) fundamental matrix sequence.*

$E = I_n$, $(zI_n - A)^{-1}$ is called the *resolvent* of A.

$(zE - A)^{-1}$ is called *generalized resolvent* [Rose, 1978].

Backward fundamental matrix of regular matrix pencils

Definition. [Langenhop, 1971] If $E, \; A \in \mathbb{C}^{n\times n}$ and $zE - A$ is invertible for some $z \in \mathbb{C}$, then for some $R_2 > 0$ and $0 < |z| < R_2$, we have the Laurent expansion at zero of

$$(zE - A)^{-1} = V_p z^{-p} + \cdots + V_0 + V_{-1} z^1 + V_{-2} z^2 + \cdots = \sum_{i=-p}^{\infty} V_{-i} z^i$$

where the coefficient matrices $V_i \in \mathbb{C}^{n\times n}$ are independent of z and are uniquely determined by E, A. $V_i, \; i = p, p-1, ..$ are known as the *backward fundamental matrix sequence.*

Computation of the fundamental matrix of a regular matrix pencil

- [Langenhop, 1971]. Computation of V_i in terms of V_0 and V_{-1}
- [Rose, 1978], [Campbell, 1980]. Computation of V_i using the Drazin inverse [Yimin Wei, 2000] of *E*, *A*.
- [Mertzios, 1989]. Leverrier-Fadeev method.

Properties of the fundamental matrix of regular matrix pencils.

Theorem. *[*Langenhop, 1971*], [Mertzios 1989] With* $(zE - A)$ *regular and* Φ_i *the forward fundamental matrix sequence the following equations hold:*

1. $\Phi_i E - \Phi_{i-1} A = I\delta_i$

2. $E\Phi_i - A\Phi_{i-1} = I\delta_i$

3. $\Phi_i = \begin{cases} (\Phi_0 A)^i \Phi_0 & i \geq 0 \\ (-\Phi_{-1} E)^{-i-1} \Phi_{-1} & i < 0 \end{cases}$

4. $\Phi_i E \Phi_j = \Phi_j E \Phi_i$

5. $\Phi_i E \Phi_j = \begin{cases} -\Phi_{i+j} & i < 0, j < 0 \\ \Phi_{i+j} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$

6. $\Phi_i A \Phi_j = \begin{cases} -\Phi_{i+j+1} & i < 0, j < 0 \\ \Phi_{i+j+1} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$

where δ_i *is the Kronecker delta.*

An application of the forward fundamental matrix sequence.

Consider the singular dynamical system of equations

$$Ex_{k+1} = Ax_k + Bu_k \quad k = 0,1,\ldots,N-1$$

with $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m$. Let the interval of interest of the index k be $[0,N]$ and the forward fundamental matrix sequence sequence of $(zE-A)^{-1}$ be the following

$$(zE-A)^{-1} = z^{-1}\sum_{i=-\mu}^{\infty}\Phi_i z^{-i}$$

[Lewis, 1990]. *Forward solution.*

Consider that the initial condition x_0 is given and that it is desired to determine the state x_k in a forward fashion from the input sequence and the previous values of the semistate.

$$x_k = \Phi_k E x_0 + \sum_{i=0}^{k+\mu-1}\Phi_{k-i-1}Bu_i$$

Another application of the forward fundamental matrix sequence.

Consider the singular dynamical system of equations

$$Ex_{k+1} = Ax_k + Bu_k \quad k = 0,1,\ldots,N-1$$

with $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m$. Let the interval of interest of the index k be $[0,N]$ and the forward fundamental matrix sequence sequence of $(zE - A)^{-1}$ be the following

$$(zE - A)^{-1} = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}$$

[Lewis, 1990]. *Symmetric solution.*

Another interpretation, arising in economics (where k might not be the time variable) and elsewhere, is to determine the semistate x_k for intermediate values of k , given the sequence $\{u_k\}$ and admissible x_0 and x_N .

$$x_k = \Phi_k E x_0 - \Phi_{-N+k} E x_N + \sum_{i=0}^{N-1} \Phi_{k-i-1} B u_i$$

An application of the backward fundamental matrix sequence.

Consider the singular dynamical system of equations

$$Ex_{k+1} = Ax_k + Bu_k \quad k = 0,1,\ldots,N-1$$

with $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m$. Let the interval of interest of the index k be $[0,N]$ and the backward fundamental matrix sequence sequence of $(zE - A)^{-1}$ be the following

$$(zE - A)^{-1} = \sum_{i=-p}^{\infty} V_{-i} z^i$$

[Lewis, 1990]. *Backward solution.*
Consider that the final condition x_N is given and then determine x_k in a backward fashion from the input and future values of the semistate.

$$x_k = -V_{k-N-1}Ex_N + \sum_{i=k-p}^{N-1} V_{k-i}Bu_i$$

Fundamental matrix of the inverse of a polynomial matrix.

Definition. Consider the polynomial matrix

$$A(z) = A_0 + A_1 z + \ldots + A_q z^q \in \mathbb{R}[z]^{r\times r}$$

a) By assuming that $A(z)$ is regular i.e. $\det A(z) \neq 0$, then for some $R_1 > 0$ and $|z| > R_1$, we have the Laurent expansion at infinity of

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r - 1} z^{\hat{q}_r - 1} + \ldots = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}$$

where the coefficient matrices $H_i \in \mathbb{C}^{n\times n}$ are independent of z and are uniquely determined by $A(z)$. $H_i,\ i = -\hat{q}_r, -\hat{q}_r + 1, ..$ are known as the *(forward) fundamental matrix sequence.*

b) By assuming that $A(z)$ is regular i.e. $\det A(z) \neq 0$, then for some $R_2 > 0$ and $0 < |z| < R_2$, we have the Laurent expansion at zero of

$$A(z)^{-1} = N_\nu z^{-\nu} + N_{\nu-1} z^{-\nu+1} + \cdots = \sum_{i=-\nu}^{\infty} N_{-i} z^{i}$$

where the coefficient matrices $N_i \in \mathbb{C}^{n\times n}$ are independent of z and are uniquely determined by $A(z)$. $N_i,\ i = \nu, \nu - 1, \ldots$ are known as the *backward fundamental matrix sequence*

First Problem

- Computation of the forward and backward fundamental matrix sequence of a the inverse of a polynomial matrix
  - [Fragulis, 1991]. Recursive solution of the forward fundamental matrix without reference to its properties.
- Properties of the forward and backward fundamental matrix sequence.
  - No properties like in the matrix pencil case have been found.

AutoRegressive Moving Average representations (ARMA)

Definition. A linear, time invariant discrete time system, described by the difference equation:

$$A_0 y_k + A_1 y_{k+1} + \cdots + A_q y_{k+q} = B_0 u_k + \cdots + B_q u_{k+q}, \ \ k = 0,1,\ldots,N-q$$

or

$$A(\sigma) y_k = B(\sigma) u_k$$

is called an ARMA representation of the system, where σ denotes the shift-forward operator, $y_k : [0,N] \rightarrow \mathbb{R}^r$ is the output of the system, $u_k : [0,N] \rightarrow \mathbb{R}^m$ is a known input of the system, and

$$A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in \mathbb{R}[\sigma]^{r\times r}, \ \ |A(\sigma)| \neq 0$$

$$B(\sigma) = B_0 + B_1\sigma + \cdots + B_q\sigma^q \in \mathbb{R}[\sigma]^{r\times m}$$

Second problem

- Solutions of ARMA Systems [Karampetakis et al, 2001]
  - The solutions have already been found in a closed form, as a result of ***cumbersome*** calculations.

First thoughts

$$A_0 y_k + A_1 y_{k+1} + \cdots + A_q y_{k+q} = 0,\ \ k = 0,1,\ldots,N-q$$

or

$$\underbrace{\left(A_0 + A_1\sigma + \cdots + A_q\sigma^q\right)}_{A(\sigma)} y_k = 0,\ \ k = 0,1,\ldots,\mathrm{N}$$

In case where $x_k = \begin{bmatrix} y_k^T & y_{k+1}^T & \cdots & y_{k+q-1}^T \end{bmatrix}^T$ then we define as

$$Ex_{k+1} + Ax_k = 0 \text{ or } (\sigma E + A)x_k = 0 \text{ with } k = 0,1,\ldots,\mathrm{N}$$

$$E = \begin{bmatrix} I_r & 0 & \ldots & 0 & 0 \\ 0 & I_r & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_r & 0 \\ 0 & 0 & \ldots & 0 & A_q \end{bmatrix} \in \mathbb{R}^{rq\times rq}, A = \begin{bmatrix} 0 & -I_r & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -I_r \\ -A_0 & -A_1 & \ldots & -A_{q-2} & -A_{q-1} \end{bmatrix} \in \mathbb{R}^{rq\times}$$

First thoughts

- Why matrix pencils ?
  - A number of recent algorithms are trying to reduce the polynomial matrix problems to matrix pencil problems since we can use more robust and reliable algorithms for their solutions.
- Why not this matrix pencil ?
  - However, the connection between the fundamental matrix sequence of $A(z)^{(-1)}$ and the one of $(zE+A)^{(-1)}$ is complicated.

Rewriting the system matrix equations

We may rewrite the system equations for $k = 0,1,..,q-1$ as follows

$$
\left[\begin{array}{cccc|cccc|cccc}
A_q & A_{q-1} & \cdots & A_1 & A_0 & 0 & \cdots & 0 & & & & \\
0 & A_q & \cdots & A_2 & A_1 & A_0 & \cdots & 0 & & 0 & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & & & \\
0 & \cdots & 0 & A_q & A_{q-1} & A_{q-2} & \cdots & A_0 & & & & \\
\hline
 & & & & A_q & A_{q-1} & \cdots & A_1 & A_0 & 0 & \cdots & 0 \\
 & 0 & & & 0 & A_q & \cdots & A_2 & A_1 & A_0 & \cdots & 0 \\
 & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & & 0 & \cdots & 0 & A_q & A_{q-1} & A_{q-2} & \cdots & A_0 \\
\hline
 & \vdots & & & & \ddots & & & & \ddots & & \ddots
\end{array}\right]
\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} =
$$

$$
= \begin{bmatrix}
B_q & \cdots & B_0 & 0 & \cdots & 0 \\
0 & B_q & \cdots & B_0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & B_q & B_{q-1} & \cdots & B_0
\end{bmatrix}
\begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}
$$

A

E

B

Rewriting the system matrix equations

$$\tilde{E}x_{k+1} + \tilde{A}x_k = \tilde{B}v_k \quad k = 0,1,\ldots,\left\lceil \frac{N}{q} \right\rceil - 1$$

where

$$\tilde{E} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \in \mathbb{R}^{qr \times qr}, \quad \tilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{qr \times qr},$$

$$\tilde{B} = \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \in \mathbb{R}^{qr \times 2qr}$$

and

$$x_k = \begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix}; \; v_k = \begin{bmatrix} u_{kq+2q-1} \\ u_{kq+2q-2} \\ \vdots \\ u_{kq+0} \end{bmatrix}$$

Relation between $A(z)$ and $\left(z\tilde{E} + \tilde{A}\right)$

- Same infinite elementary divisors
- If λ is a finite zero of $A(z)$, λ^q is a finite zero of $\left(z\tilde{E} + \tilde{A}\right)$
- The above properties ensure that stability/controllability/observability is preserved.
- The Laurent expansion of the $A(z)^{-1}$ can be easily computed through the Laurent expansion of $(zE + A)^{-1}$.

Computation of the fundamental matrix sequence of $A(z)^{-1}$

$$A(z)^{-1} = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i} = \sum_{i=-\nu}^{\infty} N_{-i} z^{i}$$

$$\left(z\tilde{E} + \tilde{A}\right)^{-1} = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i} = \sum_{i=-p}^{\infty} V_{-i} z^{i}$$

Theorem *The coefficients* H_i *(resp.* N_i*) of the Laurent series expansion at infinity (zero) of* $A(z)^{-1}$ *and those* Φ_i *(*V_i*) of* $\left(z\tilde{E} + \tilde{A}\right)^{-1}$ *are connected by:*

$$\Phi_i = H^{q,q}_{-q-qi} = \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}$$

and

$$V_i = N^{q,q}_{-qi} = \begin{bmatrix} N_{-qi} & N_{-qi-1} & \cdots & N_{-qi-q+1} \\ N_{-qi+1} & N_{-qi} & \cdots & N_{-qi-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ N_{-qi+q-1} & N_{-qi+q-2} & \cdots & N_{-qi} \end{bmatrix}$$

Computation of the fundamental matrix sequence of $A(z)^{-1}$

Proof

$$\tilde{E}\Phi_i + \tilde{A}\Phi_{i-1} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix} +$$

$$\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} H_{-qi} & H_{-qi-1} & \cdots & H_{-q-qi+1} \\ H_{-qi+1} & H_{-qi} & \cdots & H_{-q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi+q-1} & H_{-qi+q-2} & \cdots & H_{-qi} \end{bmatrix} = I_{qr}\delta_i$$

where

$$A(z)A(z)^{-1} = I_r \Leftrightarrow \left(\sum_{i=0}^{q} A_i z^i\right)\left(\sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}\right) = I_r \Leftrightarrow$$

$$\sum_{i=0}^{q} A_i H_{i-k} = \delta_k I_r \quad \left(\text{or } \sum_{i=0}^{q} H_{i-k} A_i = \delta_k I_r\right)$$

Algorithm for the computation of the fundamental matrix

Consider the inversion of the polynomial matrix

$$A(z) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} z^2$$

Step 1. *Construct the compound matrices* $\tilde{E}, \tilde{A}$ *defined previously.*

$$\tilde{E} = \begin{bmatrix} A_2 & A_1 \\ 0 & A_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \tilde{A} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Algorithm for the computation of the fundamental matrix

Step 2. *Determine the matrices* Φ_0, Φ_{-1} *of the resolvent* $\left(z\tilde{E} + \tilde{A}\right)^{-1}$ *using one of the known computing techniques described in [Campbell], [Dziurla], [Mertzios], [Rose] and [Schweitzer], and therefore compute the coefficients* $H_{-2q+1}, \ldots, H_{q-1}$ *from the following relations*

$$\Phi_0 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix}, \quad \Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}$$

$$\Phi_0 = \begin{bmatrix} H_{-2} & H_{-3} \\ H_{-1} & H_{-2} \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore

$$H_{-3} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}, H_{-2} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, H_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, H_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Algorithm for the computation of the fundamental matrix

Step 3. *The rest terms can be determined by using the following equations*

$$\Phi_i = \left\{ \begin{array}{ll} \left(-\Phi_0 \tilde{A}\right)^i \Phi_0 & i \geq 0 \\ \left(-\Phi_{-1} \tilde{E}\right)^{-i-1} \Phi_{-1} & i < 0 \end{array} \right\} = \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}$$

Properties of the forward fundamental matrix sequence

$$\Phi_i = H_{-q-qi}^{q,q} = \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix},$$

$$\left(z\tilde{E} + \tilde{A}\right)^{-1} = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}$$

$$\tilde{E} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \in \mathbb{R}^{qr \times qr}, \quad \tilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{qr \times qr},$$

Properties of the forward fundamental matrix sequence

Theorem. *With* $A(z)$ *regular and* Φ_i *defined earlier we can prove the following equations:*

1. $\Phi_i \tilde{E} + \Phi_{i-1}\tilde{A} = I\delta_i$
2. $\tilde{E}\Phi_i + \tilde{A}\Phi_{i-1} = I\delta_i$
3. $\Phi_i = \begin{cases} \left(-\Phi_0 \tilde{A}\right)^i \Phi_0 & i \geq 0 \\ \left(-\Phi_{-1}\tilde{E}\right)^{-i-1} \Phi_{-1} & i < 0 \end{cases}$
4. $\Phi_i \tilde{E} \Phi_j = \Phi_j \tilde{E} \Phi_i$
5. $\Phi_i \tilde{E} \Phi_j = \begin{cases} -\Phi_{i+j} & i < 0, j < 0 \\ \Phi_{i+j} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$
6. $\Phi_i \tilde{A} \Phi_j = \begin{cases} -\Phi_{i+j+1} & i < 0, j < 0 \\ \Phi_{i+j+1} & i \geq 0, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$

The above properties will be crucial to the analysis of ARMA representations.

Applications to difference equations - forward solution

Theorem *The forward solution of ARMA is the following:*

$$y_k = \begin{bmatrix} H_{-k-q} & H_{-k-q+1} & \cdots & H_{-k-1} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} +$$

$$\begin{bmatrix} H_{-k} & H_{-k+1} & \cdots & H_{\hat{q}_r} \end{bmatrix} \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k+\hat{q}_r+q} \end{bmatrix}$$

Proof

- *Solution of the corresponding descriptor system*

$$x_k = \Phi_k \tilde{E} x_0 + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} \tilde{B} u_i$$

- *Using the connections between the fundamental matrices and their properties, we arrive at the above equations.*

$$\Phi_k = \begin{bmatrix} H_{-q-qk} & H_{-q-qk-1} & \cdots & H_{-2q-qk+1} \\ H_{-q-qk+1} & H_{-q-qk} & \cdots & H_{-2q-qk+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qk-1} & H_{-qk-2} & \cdots & H_{-qk-q} \end{bmatrix}; x_k = \begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix}; u_i = \begin{bmatrix} u_{iq+2q-1} \\ u_{iq+2q-2} \\ \vdots \\ u_{iq+0} \end{bmatrix}$$

Applications to difference equations- backward solution

Theorem *The backward solution of ARMA is the following:*

$$y_k = \begin{bmatrix} N_{N-k} & N_{N-k-1} & \cdots & N_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +$$

$$\begin{bmatrix} N_{N-k-q} & N_{N-k-q-1} & \cdots & N_{-p} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{k-p} \end{bmatrix}$$

Applications to difference equations- symmetric solution

Theorem *The symmetric solution of ARMA is the following:*

$$y_k = \begin{bmatrix} H_{-k-1} & H_{-k-2} & \cdots & H_{-k-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} +$$

$$\begin{bmatrix} H_{N-k} & H_{N-k-1} & \cdots & H_{N-q-k+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +$$

$$\begin{bmatrix} H_{-q+N-k} & H_{-q+N-k-1} & \cdots & H_{-k} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Conclusions.

- **Computation of the fundamental matrix**
  - *Advantages*

    The problem has been reduced to a matrix pencil problem that can be solved by robust and reliable algorithms
  - *Disadvantages*

    Matrices of larger dimensions are used.
- Several **new properties for the fundamental matrices of the inverse of a polynomial matrix** have been found similar to [Langenhop, 1971]
- The **forward, backward and symmetric solutions of ARMA equations** have been found through this reduction technique.
- **Further research**
  - Definition of controllability, observability and reachability for ARMA representations.
  - Tests for controllability, observability and reachability of ARMA representations.

Further research – symmetric reachability

Theorem *The descriptor symmetric reachability matrix over the interval* $[0, N]$ *is defined as*

$$R_S(N) = \left[(\Phi_0 + \Phi_{-N})B \quad (\Phi_1 + \Phi_{-N+1})B \quad \cdots \quad (\Phi_{N-1} + \Phi_{-1})B \right]$$

Then the descriptor system is reachable iff

$$rank(R_S(N)) = r$$